Abstract— Two deterministic second order ordinary differential equations and their stochastic analogues are presented. The first equation describes the concentration of charge in the capacitor of a simple L-R-C circuit and can be solved explicitly in both the deterministic and the stochastic case. The second one is the well-known Van der Pol oscillator, which belongs in the class of relaxation oscillators and arises in circuits of a more complex structure. It can only be approximated in both the deterministic and the stochastic case. The distributional properties of the solutions or approximations of both stochastic differential equations are explored and a graphical representation of the solutions is done as well. For the approximations of the stochastic differential equations a numerical scheme recently presented in literature is applied.

Keywords— Relaxation Oscillations, Van der Pol equation, Stochastic Numerical Approximations, Stochastic Differential Equations

I. INTRODUCTION

During the past fifty years, interest in the study of stochastic phenomena has increased dramatically. Intensified research activity in this area has been stimulated by the need to take into account random effects in complicated physical systems. For such systems, Markov processes have provided a natural replacement for deterministic functions as mathematical descriptions of state. Consequently, a number of methods for constructing these processes have been introduced. One such method that is well known and extensively applied obtains the distribution of the process from Kolmogorov’s equations; however these partial differential equations for the transition density of the process can be difficult to solve.

Another technique of particular relevance when the random effects can be considered as external to some reasonably well understood deterministic system replaces input parameters in the deterministic model by random processes; the resulting random model characterizes the sample path structure of the solution process. An example that has had wide application in engineering consists of adding a noise term to the right side of a deterministic equation. Langevin formulated the first such model in 1908 to describe the velocity of a particle moving in a random force field. Random differential equations of this type can be interpreted as stochastic differential equations, following Ito’s basic work in the early 1940s. Solutions of such equations represent Markov diffusion processes, the prototype of which is the Brownian motion process alternatively called Wiener process. The theory of stochastic differential equations set down by Ito [4], and independently established in the Soviet Union by Gikhman, together with the previous mathematical work of Wiener and Levy on Brownian motion has provided the basic tools making this more ambitious approach of constructing sample paths feasible.

In this paper, the properties of the explicit solutions (when possible) or approximations of the deterministic and stochastic versions of two second-order ordinary differential equations are explored. The equations presented in section I, model the concentration of charge at a fixed point of a simple L-R-C series circuit in the presence of deterministic and stochastic electromotive source. For the stochastic case, the distributional properties of the solution and an efficient numerical scheme along with its statistical properties are presented. In section II, analogous results are presented for the Van der Pol oscillator that models the concentration of charge in circuits of a more complex structure; for the simulation of its trajectory only approximations can be done in both the deterministic and the stochastic case.
II. DETERMINISTIC AND STOCHASTIC DIFFERENTIAL EQUATION MODELLING FOR SIMPLE ELECTRICAL NETWORKS

A. Genesis scheme and general solution for the deterministic case

Let \( Q(t) \) be the charge on the capacitor of a simple L-R-C series circuit. Let also \( F(t) \) denote the electromotive source provided to the system at time \( t \) (this may be a battery or a generator which produces a potential difference and causes a current \( I \) to flow through the circuit). \( R \) represents a resistance to the flow of the current such as that produced by a light bulb or a toaster. When current flows through a coil of wire \( L \), a magnetic field is produced which opposes any change in the current through the coil. The change in voltage produced by the coil is proportional to the rate of change of the current and the constant of proportionality is called the inductance \( L \) of the coil. A capacitor, or condenser, indicated by \( C \), usually consists of two metal plates separated by a material through which very little current can flow; the flow of the current is reversed as one plate or the other becomes fully charged.

To derive a differential equation, which is satisfied by \( Q(t) \), one may use the Kirchhoff’s second law which states that the impressed voltage in a closed circuit equals the sum of the voltage drops in the rest of the circuit. By Ohm’s Law the voltage drop across a resistance equals \( R \cdot I \). The voltage drop across the inductance of \( L \) henrys equals \( L \frac{dI}{dt} \) and across the capacitance of \( C \) farads equals \( \frac{Q}{C} \). Hence

\[
F(t) = L \frac{dI}{dt} + R \cdot I + \frac{Q}{C},
\]

and since \( I = \frac{dQ(t)}{dt} \), the following equation holds

\[
L \cdot \frac{d^2Q(t)}{dt^2} + R \cdot \frac{dQ(t)}{dt} + \frac{C}{Q(t)} = F(t) \quad (1.1)
\]

For the derivation of the solution of this second order ordinary differential equation a number of methods may be applied depending on the type of the electromotive source function. When \( F(t) \) is piecewise continuous with a finite number of maximums and minimums and \( \int_{-\infty}^{\infty} |F(t)|^2 \, dt < \infty \) holds, Fourier transforms may be applied.

The Fourier transform of \( F(t) \) is

\[
\tilde{F}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\omega t} F(t) \, dt.
\]

Suppose also that

\[
\tilde{Q}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \frac{dQ(t)}{dt} e^{i\omega t} \, dt
\]

is the Fourier transform of the specific solution of the non-homogeneous differential equation. Then it holds that

\[
\tilde{Q}(\omega) = i\omega \tilde{F}(\omega)
\]

or

\[
\tilde{Q}_s(\omega) = \frac{-i\omega}{-\omega^2 L - i\omega R + C^{-1}} \tilde{F}(\omega) = \frac{\tilde{F}(\omega)}{Z(\omega)}
\]

where \( Z(\omega) = R - i \left( \frac{\omega L}{\omega C} \right) \) is called the impedance of the circuit. Hence

\[
Q_s(t) = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{\tilde{F}(\omega) e^{-i\omega t}}{Z(\omega)} \, d\omega
\]

is the non-transient solution.

B. Genesis scheme and solution for the stochastic case

The potential source may not be deterministic but of the form

\[
F(t) = G(t) + \text{"noise"}
\]

as proposed by Oksendal [11]. The equation obtained by allowing randomness in the coefficients of a differential equation is a stochastic differential equation. Here, the case where the noise is described by a Wiener process is discussed. Equation (1.1) is being modified to give

\[
LQ''_s(t) + RQ'_s(t) + C^{-1}Q_s(t) = G(t) + aW_t
\]

(1.2) where \( a \) is called the intensity of the noise and its magnitude determines the deviation of the stochastic case from the deterministic one. By introducing the vector

\[
X = X(t, \omega) = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} = \begin{bmatrix} Q_s \\ Q'_s \end{bmatrix}
\]

(1.2) may be transformed to give

\[
\begin{bmatrix} X'_1 \\ X'_2 \end{bmatrix} = \begin{bmatrix} -RX_2 - C^{-1}X_1 + G_i + aW_t \\ 0 \end{bmatrix}
\]

or in matrix notation

\[
dX = AXdt + Hdt + KdW_t
\]

(1.3)

where

\[
dX = \begin{bmatrix} dX_1 \\ dX_2 \end{bmatrix}, A = \begin{bmatrix} 0 & 1 \\ (-CL)^{-1} & -RL^{-1} \end{bmatrix}, H = \begin{bmatrix} 0 \\ L^{-1}G_i \end{bmatrix}, K = \begin{bmatrix} 0 \\ aL^{-1} \end{bmatrix}
\]

and \( W_t \) is a one-dimensional Wiener process. Equation (1.3) represents a 2-dimensional stochastic differential equation and may be written as

\[
\exp(-At)dX = \exp(-At)AXdt + \exp(-At)Hdt + KdW_t
\]

(1.4)

where for a general \( n \times n \) matrix \( F \), \( \exp(F) = \sum_{n=0}^{\infty} \frac{F^n}{n!} \) ; the series converge for every \( F \) and its summation has the properties

\[
(e^{A_1}e^{A_2}) = e^{A_1+A_2}, \quad (e^{A})^n = I, \quad \frac{d}{ds} (e^{As}) = Ae^{As}.
\]

When \( F \) can be written as \( F = JAS^{-1} \) the differential equation

\[
\frac{du}{dt} = Fu
\]

may be solved to give

\[
u(t) = e^{pt}u_0 = Se^{\Lambda t}S^{-1}u_0.
\]
The columns of $\mathbf{S}$ are the eigenvectors of $F$ so

$$
\mathbf{u}(t) = \begin{bmatrix} e^{\lambda_1 t} & \ldots & 0 \\
0 & \ldots & 0 \\
0 & \ldots & e^{\lambda_n t}
\end{bmatrix} \mathbf{S}^{-1} \mathbf{u}_0 = c_1 e^{\lambda_1 t} x_1 + \ldots + c_n e^{\lambda_n t} x_n.
$$

The general solution is a combination of exponential functions and the constants $c_i$ are given by the relation $c = \mathbf{S}^{-1} \mathbf{u}_0$. Here it is tempting to relate the left hand side of (1.4) to $d(\exp(-At))X$. To do this we use the 2-dimensional Ito formula (for a thorough discussion on Ito’s formula see Oksendal [11]). Applying this result to the two coordinate functions $g_1, g_2$ of $g : [0, \infty) \times \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$
g(t, x_1, x_2) = \exp(-At) \begin{bmatrix} x_1 \\
x_2
\end{bmatrix}
$$

we obtain that

$$
d(\exp(-At)) = (-At) \exp(-At) X dt + \exp(-At) dX.
$$

Substituted in (1.4) this gives

$$
\exp(-At)X - X_0 = \int_0^t \exp(-As) H_s ds + \int_0^t \exp(-As) K dW_s,
$$

where the last integral is an Ito integral (see Ikeda and Watanabe). Now, the integration by parts theorem (Oksendal [11]) may be applied to give

$$
X = \exp(At) \left[ X_0 + \int_0^t \exp(-As) H_s ds + \int_0^t \exp(-As) K dW_s \right].
$$

C. Probabilistic properties and numerical approximations for the stochastic solution

Let an equation of the form

$$
x'' + 2\delta x' + \omega_0^2 x = \xi(t)
$$

represent a linear oscillator that is subject to a random action $\xi(t)$. Solving (1.5) with arbitrary initial conditions $x = x_0, x' = x_0'$ for $t = t_0$, we obtain

$$
x(t) = e^{-\delta(t-t_0)} \left[ x_0 \cos(\Omega(t-t_0)) + \frac{x_0' + \delta x_0}{\Omega} \sin(\Omega(t-t_0)) + \right]
$$

$$
+ \frac{1}{\Omega} \int_{t_0}^t e^{-\delta(t-\tau)} \xi(\tau) \sin(\Omega(t-\tau)) d\tau
$$

(1.6)

where $\Omega^2 = \omega_0^2 - \delta^2$. Let $\delta > 0$. Then it follows from (1.6) that

$$
x(t) = \frac{1}{\Omega} \int_{t_0}^t e^{-\delta(t-\tau)} \xi(\tau) \sin(\Omega(t-\tau)) d\tau
$$

as $t_0 \to \infty$. According to (1.6) the random action $\xi(t)$ is associated with the random change $x(t)$. However, if $\xi(t)$ is a stationary random process with zero expectation and given autocorrelation function $K_{\xi}(t)$ then $x(t)$ is also a stationary random process with zero expectation and with autocorrelation function

$$
K_{x}(\tau) = \frac{1}{\Omega} \int_{0}^{\tau} e^{-\delta(t+\tau)} \sin(\Omega(t+\Omega)) d\Omega d\mu
$$

(see Neimark and Landa [10]). The relationship between the spectral density $S_{\xi}(\omega)$ of the random action $\xi(t)$ and the spectral density $S_{x}(\omega)$, associated with the equation presented above, is known to be of the form

$$
S_{\xi}(\omega) = \frac{S_{x}(\omega)}{(\omega^2 - \omega_0^2)^2 + 4\delta^2 \omega^2}.
$$

Thus a linear oscillator with an external random action $\xi(t)$ can be regarded as a transformer of $\xi(t)$ into a new random process $x(t)$, while (1.6), (1.7) establish the relationships between the autocorrelation functions and the spectral densities in the case where $\xi(t)$ is a stationary random process, enabling as to find the variance of the random variable $x(t)$

$$
Var(x) = K_{x}(0) = \frac{1}{\Omega} \int_{0}^{\tau} e^{-\delta(t-\tau)} \xi(\tau) \sin(\Omega(t-\Omega)) d\Omega d\mu
$$

$$
= \frac{1}{\pi} \int_{-\infty}^{\infty} S_{\xi}(\omega) d\omega
$$

The vanishing of $Var(\xi)$ entails that of $Var(x)$ as expected. It follows from $Var(\xi)=0$ that $K_{x}(\tau)=0$, i.e., if the action $\xi(t)$ is not random, then the thereby generated change in the coordinate $x(t)$ of the linear oscillator is not random. If the random action $\xi(t)$ is replaced by the sum of a certain given action $\xi_0(t)$ and the same random action $\xi(t)$, then if the random component of $\xi(t)$ vanishes, the output quantity $x(t)$ does not; however, it loses its randomness and becomes deterministic; i.e., as expected, the randomness of $x(t)$ is only due to that of $\xi(t)$.

The above case illustrates the presence of “stochasticity transformers” in the dynamical systems of linear oscillators. Meanwhile, external noise is transformed into random changes of $x, x'$. Randonness in dynamical systems has only been considered in this respect until quite recently. Randomness was acquired by a dynamical system from the outside in the form of a certain random action. It is this action that caused the randomness of change in the state of the system and in many other quantities determined, which enabled us to represent a dynamical system as a certain transformer of random inputs into random outputs.

In order to approximate numerically the solution of equation (1.2) one may take advantage of the structure of equation (1.1) to obtain a relatively simple multi-step scheme. Multi-step methods are often more efficient computationally than one-step methods of the same order of strong convergence because they require essentially only one new evaluation of the right hand side of the differential equation for each iteration as mentioned by Milstein [8]. A one step scheme should be used though, to generate the initial steps needed to start the multi-step scheme.
The 2-dimensional Ito system (1.3) may be approached as a specific case of the system
\[
\begin{align*}
\frac{dX_1^i}{dt} &= X_2^i\Delta t \\
\frac{dX_2^i}{dt} &= \left[-a(t)X_2^i + b(t,X_1^i)\right]dt + \sum_{j=1}^{m}c_j(t,X_1^i)dW_j^i
\end{align*}
\] (1.9)
The Milstein scheme for the system (1.9) takes the form
\[
\begin{align*}
Y_{n+1}^1 &= Y_n^1 + Y_n^2\Delta \\
Y_{n+1}^2 &= a(t,Y_n^2)\Delta + b(t,Y_n^2)\Delta + \frac{1}{2}c(t,Y_n^2)\Delta dW^i
\end{align*}
\] (1.10)
and has strong order of convergence 1.0, that is the proposed stochastic numerical scheme is equivalent to the deterministic Euler scheme. \(Y_n^i\) denotes the nth step of the approximation; \(\Delta\) is equal to the length of the (time) discretization interval and \(\Delta W^i\) denotes the increment of the Wiener process at the corresponding time interval. For more information on stochastic approximations the reader may consult the books written by Kloeden & Platen [5] and Kloeden, Platen & Schurz [6]. A useful simplifying feature of (1.10) is the absence of double Ito integrals. Moreover, one can solve the first equation for
\[
Y_n^2 = \frac{1}{\Delta}(Y_{n+1}^1 - Y_n^1)
\]
and insert it to the second to obtain a two-step scheme for the first component \(Y_1^i\), provided we use an equidistant discretization of the time interval. The resulting two-step scheme
\[
\begin{align*}
Y_{n+1}^1 &= 2 - a(t)\Delta Y_n^1 - \left[1-a(t)\Delta\right]Y_n^1 + b(t,Y_n^1)\Delta \\
&\quad + \sum_{j=1}^{m}c_j(t,Y_n^1)\Delta dW_j^i
\end{align*}
\] (1.11)
is due to Lepingle and Ribemont [7]. The first equation of (1.10) can be used both as a starting routine for (1.11), and to calculate approximations of the second component if they are required. The Lepingle-Ribemont scheme was applied to equation (1.9) for \(L=1, R=4, C=1\) and various source functions and intensities. Some of the results are depicted right below

A second type of problems does not require a good pathwise approximation. For instance, one may only be interested in the computation of moments or other functionals of the Ito process; the requirements for their simulation are not as demanding as for pathwise approximations. Some computer experiments were carried out to investigate the mean error
\[
\mu = E(Y(T)) - E(X_f)
\]
where \(X_f\) denotes the explicit solution of (1.9) and \(Y(T)\) its Lepingle-Ribemont approximation. The simulated trajectories were arranged into \(M\) batches with \(N\) trajectories each. The mean error of the jth batch is estimated by the statistic
\[
\hat{\mu}_j = \frac{1}{N}\sum_{i=1}^{N}Y_{T,h,j} - E(X_f)
\]
for \(j=1,2,\ldots,M\), and their average by the statistic
\[
\hat{\mu} = \frac{1}{M}\sum_{j=1}^{M}\hat{\mu}_j = \frac{1}{MN}\sum_{j=1}^{M}\sum_{i=1}^{N}Y_{T,h,j} - E(X_f)
\] .
Similarly, the variance of the batch averages \(\hat{\mu}_j\) is given by
\[
\hat{\sigma}_\mu^2 = \frac{1}{M-1}\sum_{j=1}^{M}(\hat{\mu}_j - \hat{\mu})^2
\]
and the \(100(1-\alpha)\%\) confidence interval of the student-t distribution with \(M-1\) degrees of freedom for the mean error \(\mu\) is
\[
\left[\hat{\mu} - t_{1-\alpha/2,M-1}\sqrt{\frac{\hat{\sigma}_\mu^2}{M}}, \hat{\mu} + t_{1-\alpha/2,M-1}\sqrt{\frac{\hat{\sigma}_\mu^2}{M}}\right]
\]
The mean error \(\mu\) will lie in this confidence interval with at least probability 1-\(\alpha\). The figures presented right below correspond to \(M=10\) batches of \(N=1000\) trajectories of the Lepingle-Ribemont approximation for the Ito process (1.9) with \(X_0 = 0,1, X_0 = 0\), \(L = 1, R = 4, C = 1, F(t) = 5\), \(a = 0.1\), for step length \(\Delta=10^{-2}\) and terminal time \(T=10\). The 90% confidence intervals for the mean error \(\mu\) were determined and the process was repeated for \(M=20, 40\) and 100 batches using the batches already simulated and plot the intervals on \(\mu\) versus \(M\) axes.
Figure 1.4: Confidence intervals for increasing batch size

$M=20$ batches of $N=1000$ trajectories of the Lepingle-Ribemont approximation were generated as in the previous case. The 90% confidence intervals for the mean error $\mu$ for step sizes $\Delta=1/40, 1/80, 1/160$ and $1/320$ were calculated and plotted on $\mu$ versus $\Delta$ axes.

Figure 1.5: Confidence intervals for increasing time step size

When $\Delta$ is very small the variance may be again large due to roundoff errors and can destroy the result. Nevertheless, it appears that $\hat{\mu}$ depends strongly on $\Delta$. It’s useful here to plot the results in log$_2$ versus log$_2$ coordinates. We shall also call $[\mu]$ the mean error for convenience.

Figure 1.6: log$_2$ of the mean error versus log$_2$$\Delta$

THE VAN DER POL OSCILLATOR

In various fields of science, notably in physics and biology, one is confronted with periodic phenomena having a remarkable temporal structure: it is as if certain systems are periodically reset in an initial state. A paper of Van der Pol [12] in the Philosophical Magazine of 1926 started up the investigation of this highly nonlinear type of oscillation for which Van der Pol coined the name “relaxation oscillation”. Van der Pol studied properties of a triode circuit that exhibited self-sustained oscillations with amplitude independent of the starting conditions. For certain values of the system parameters the oscillation is almost sinusoidal, while in a different range the system shows abrupt changes, see figures 2.2, 2.3, 2.4. In the last case the period turns out to be proportional to a large parameter of the system. The name “relaxation oscillation” refers to this characteristic time of the system. In figure 2.1 the triode circuit is given. The system satisfies the following equation

$$L \frac{dI}{dt} + RI + V = M \frac{dI_s}{dt}$$  \hspace{1cm} (2.1)

where $L$ is the selfinductance, $M$ is the mutual inductance and $I$ and $V$ respectively, a current and a grid voltage. Assuming that the grid current is negligible we have $I = CV'(t)$, where $C$ is a capacitance. The triode characteristic is idealized as $I_a = V = \frac{1}{3} V^3$. Then by substituting

$$V = x\sqrt{1 - RCM^{-1}} \quad t = \tau \sqrt{LC}$$

we obtain the well-known Van der Pol equation

$$\frac{d^2x}{d\tau^2} + \nu(x^2 - 1) \frac{dx}{d\tau} + x = 0$$  \hspace{1cm} (2.2)

with

$$\nu = \frac{M}{LC} - R \sqrt{\frac{C}{L}}$$

Van der Pol investigated equation 2.2 asymptotically for $0<\nu<<1$ and gave numerical solutions for the phase plane as well as the physical plane for three characteristic choices of $\nu$. For $\nu>>1$ the periodic solution of 2.2 behaves as a relaxation oscillation.

Figure 2.1: A triode circuit exhibiting self-sustained oscillation.

Figure 2.2: The Van der Pol oscillator for $\nu=1$
Figure 2.3: The Van der Pol oscillator for $v=0.1$

Figure 2.4: The Van der Pol oscillator for $v=8$

The analytical method of Cartwright for the deterministic case

Cartwright analyzed the Van der Pol oscillator

$$\frac{d^2x}{dt^2} + v(x^2 - 1)\frac{dx}{dt} + x = 0, v \gg 1 \tag{2.3}$$

in the $x,t$-plane. Her method is based on locally valid estimates of $x$ and its derivatives, which are obtained from the following theorem proven in Cartwright [1]

For the function $x(t)$ satisfying (2.3) with $x(0) = x_0$, $x'(0) = x'_0$, a constant $M_0$ independent of $x_0$ and $x'_0$ exists such that

$$|x(t)| < M_0 t, \quad |x'(t)| < M_0 v$$

for $t > t_0(M_0, x_0, x'_0)$.

In addition to (2.3) Cartwright considered the integrated equation

$$x'(t) = x'_0 = v \left( x^2 - x_0 + \frac{x_0^3}{3} \right)$$

the energy equation

$$\left( \frac{dx}{dt} \right)^2 - (x'_0)^2 = 2v \int_{x_0}^{x} \left( x^2 - x_0 + \frac{x_0^3}{3} \right) dx$$

as well as the differentiated equation

$$\frac{d^3x}{dt^3} + v(x^2 - 1)\frac{d^2x}{dt^2} + 2v \frac{dx}{dt} \frac{dx}{dt} + \frac{dx}{dt} = 0.$$

The value of $x(t)$ and its derivative in $A$ is denoted by $a, a'$, etc., see figure 2.5. It is noted that $h' = 0, c = 1, a = 1$. In a sequence of 12 lemmas $x, x', x''$ and $t$ are estimated in the different intervals. Roughly the line of argument is as follows. In lemma 1 the energy equation is used to prove that a solution, starting in the maximum $H$ not far above $x=1$, arrives in $C$ with $c' = 1$ bounded by a number that only depends on $h$. In lemma 2 lower bounds for $x'(t)$ and $t$ are given for the interval $HC$. Lemma’s 3, 4, and 5 show that for the intervals $HY, YZ$ and $ZE$ the lower bounds given by lemma 2 are indeed good approximations of $x'(t)$ and $t$ for $h > 3/2$ and $x(t)$ not close to 1. Lemmas 6 and 7 deal, respectively with the arcs $EC$ and $CF$ just before and after $C$. They prove that the time needed to pass the strip $|x - c| < \Delta v^{2/3} < 1$ is at most $M_0^2 v^{-1/3}$ for every $\Delta > \Delta_0$ and $v > v_0(\Delta)$ with $M$ fixed. Lemma 8 gives an estimate of the time needed to reach $A'$ from $C$. As an illustration of the method, lemma 9 is stated.

A solution of 2.3 that starts in $A$ with $a' > 3/2 \mu$ reaches

$$B \text{ with } |b' - a' - \frac{2}{3} v| < t_{ab} < M \frac{\ln v}{v}.$$

Lemmas 10, 11 and 12 give estimates for the segment $BH$ for $b'$ large. The pieces are put together in the following theorem.

For the periodic solution of 2.3 the points $A, B, C$ and $H$ and the period $T$ satisfy:

$$\frac{M}{\Delta v^{2/3}} \leq |k| \leq M \Delta^{1/2} v^{-1/3}, \quad \left| a' - \frac{2}{3} v \right| \leq M \Delta^{1/2} v^{-1/3},$$

$$|b' - \frac{4}{3} v| \leq M \Delta^{1/2} v^{-1/3}, \quad h - 2 < M \Delta^{1/2} v^{-1/3} \quad \text{and}$$

$$T - v \left( \frac{3}{2} - \ln 2 \right) < M \Delta^2 v^{-1/3} \quad \text{for } \Delta > \Delta_0 \quad \text{and}$$

$$\mu > \mu_0(\Delta).$$

Later in the article, Cartwright [1] shows that $\Delta$ can be replaced by $\mu$ with $-\mu$ being the first zero of the Airy function.

Figure 2.5: The solution of the Van der Pol oscillator and the successive points between which, different points of estimates are made in the method of Cartwright.

For $0 < v < 1$ regular perturbation techniques can be applied, see Minorsky [9].
D. Numerical approximations for the stochastic case

The stochastic Van der Pol oscillator

\[ \frac{d^2 x}{dt^2} + \nu (x^2 - 1) \frac{dx}{dt} + x = a dW_t \quad (2.4) \]

may be written as 2-dimensional Ito system

\[
\begin{cases}
    dX_1^i = X_2^i \, dt \\
    dX_2^i = -\nu \left( X_1^i \right) X_2^i \, dt - X_1^i + a \, dW_t,
\end{cases}
\]

for which the Milstein scheme takes the form

\[
\begin{cases}
    Y_{n+1}^1 = Y_n^1 + Y_n^2 \Delta \\
    Y_{n+1}^2 = Y_n^2 - \nu \left( Y_n^1 \right)^2 \Delta - Y_n^1 \Delta + a \Delta W_n
\end{cases}
\quad (2.5)
\]

and has strong order 1.0. A useful simplifying feature of (2.5) is the absence of double Ito integrals. Moreover the first equation may be transformed to give

\[ Y_n^2 = \frac{1}{\Delta} \left( Y_n^1 - Y_{n-1}^1 \right) \]

and insert it into the second to obtain a two-step scheme for the first component \( Y_n^1 \), provided an equidistant discretization is used. The result is a two-step scheme

\[ Y_{n+2}^1 = \frac{1}{2} \left( \nu \left( Y_{n+1}^1 \right)^2 - 1 \right) \Delta Y_{n+1}^1 - \frac{1}{2} \nu \left( Y_n^1 \right)^2 \Delta Y_n^1 + Y_n^2 \Delta + a \Delta W_n \Delta. \]

As in section C of chapter 1 a computer experiment was conducted for the computation of the 90% confidence intervals of the numerical approximation for increasing batch size. In this case, each batch contained \( N=100 \) trajectories of the two-step Milstein approximation for the Ito process (2.4).

Figure 2.8: Confidence intervals for the mean error versus batch size

\( M=20 \) batches of \( N=100 \) trajectories of the Lepingle-Ribemont approximation were generated and the 90% confidence intervals for the mean error were calculated for step sizes of magnitude \( \Delta=1/20,1/40,1/80 \) and 1/160.

Figure 2.9: Confidence intervals for the mean error for increasing time step size

For the distributional properties of the stochastic Van der Pol oscillator we refer to Grasman [2].
Nowadays, stochastic dynamical systems take the place of the deterministic ones, when modeling of physical or mechanical processes is of concern. As expected though, the derivation of the explicit solutions (when possible) or the construction of efficient numerical schemes for the approximation of the trajectories, are more complex procedures compared to the ones of the deterministic cases. The two examples presented in this paper support this argument.

REFERENCES


