Subsidence diffusion–convection. II. The inverse problem

E. Vairaktaris a,*, I. Vardoulakis a, E. Papamichos b, V. Dougalis c

a Faculty of Applied Mathematics and Physical Sciences, Department of Mechanics, NTU of Athens, 9 Iroon Polytechnio, Zographou, Athens 157 80, Greece
b SINTEF Petroleum Research, Trondheim, Norway
c Department of Mathematics, University of Athens, Athens, Greece

Received 14 March 2003; received in revised form 16 September 2003; accepted 3 October 2003

Abstract

Considering the results of Part I of this study concerning the direct subsidence diffusion–convection (DSDC) problem we present in this paper the inverse SDC (ISDC) problem. For the regularization of the original ill-posed problem we use and compare two kinds of regularization proposals: Lions $u^{-n}$-method and the presently proposed $u^{-n}$-method. Stability in the sense of the von Neumann condition is ensured and a first approach to convergence is done in the sense of the norm of the amplification factor. Another convergence study is given in terms of the truncation error.

© 2004 Elsevier B.V. All rights reserved.

Keywords: Subsidence diffusion; Convection

1. Introduction

As is shown in Part I [1] of this study, large-scale subsidence over a yielding underground geostucture can be modeled as a stochastic (Markov) process [2]. This approach leads to the Einstein–Kolmogorov (E-K) integral equation. Under certain physical conditions the E-K integral equation satisfies a certain partial differential equation of parabolic type (Fig. 1):

$$\frac{\partial w}{\partial z} = C \frac{\partial^2 w}{\partial x^2}. \tag{1}$$

As “initial” condition for this problem serves the base subsidence and the solution yields the surface subsidence. The solution depends on the inclination angle $\beta$ of the shear-bands, which bound the yielding domain and on a dimensionless diffusivity coefficient $\hat{c}$, which determines the amount of subsidence upwards-diffusion inside the yielding domain. This constitutes the direct subsidence diffusion–convection (DSDC) problem:
\[
\frac{\partial \omega}{\partial \xi} = \frac{\hat{c}}{(1-s)^2} \frac{\partial^2 \omega}{\partial \xi^2} - \frac{\xi}{1-s} \frac{\partial \omega}{\partial \xi},
\]

which is integrated in the domain

\[0 \leq \xi \leq 1; \quad 0 < s \leq b\]

and obeys the following initial and boundary conditions

\[\omega(\xi, 0) = \frac{w_0}{B/2} \quad \text{(i.c.)},\]

\[\omega(1, s) = 0 \quad \text{(b.c.)},\]

and the symmetry condition

\[\frac{\partial \omega}{\partial \xi} \bigg|_{0,s} = 0 \quad \text{(b.c.)},\]

where the dimensionless independent space variables \(\xi, s\) are related to \(x, z\) with the following expressions

\[s = \frac{b}{H} z, \quad \xi = \frac{x}{2L(z)},\]

where

\[L(z) = \frac{B}{2} - z \cot(\beta)\]

and

\[b = 2H* \cot(\beta) > 0 \quad (0^\circ < \beta < 90^\circ), \quad H* = \frac{H}{B}.\]
The dimensionless vertical displacement \( \omega(\xi, s) \) with respect to \( w(x, z) \) is given by

\[
\omega = \frac{w}{B/2}.
\]  

(10)

The diffusivity coefficient in the original governing equation (1) is given in terms of the geometric characteristics of the subsidence mechanism and a dimensionless parameter \( \hat{c} \)

\[
C = \frac{\hat{c} B}{2} \cot(\beta).
\]  

(11)

The difficulty in large-scale problems lies however in the fact that for given surface subsidence the corresponding base displacement is not known. The problem of estimating the base displacement using the surface subsidence as “initial” conditions corresponds to the solution of an inverse SDC (ISDC) problem. Inverse diffusion problems are in general mathematically ill-posed, which means that existence, uniqueness and stability of the solution cannot be ensured. Several regularization methods have been developed for this kind of problems. Due to the dual nature (diffusion–convection) of the present problem we consider here mainly two methods of regularization for the ISDC. The first regularization scheme is essentially motivated by Lions’ method of quasi-reversibility [3]. In particular Lion’s \( u_{xxxx} \)-method is compared to the presently proposed \( u_{xx} \)-method. Stability, in the sense of the von Neumann stability condition is ensured, where the amplification factor depends on the regularization parameters. A first approach to convergence is made in the sense of the norm of the amplification factor. Another convergence study is given in terms of the truncation error [4].

2. Lion’s \( u_{xxxx} \)-regularization method

As initial condition for the ISDC problem we use the results of the corresponding DSDC, as presented in Part I for Test-2. For inverse parabolic problems Lions’ method of quasi-reversibility suggests the use of the fourth-order derivative in \( \xi \), as a regularization term. The initial, boundary-value problem is described below by Eqs. (12)–(17), where with \( \omega(\xi, s) \) denotes the dimensionless solution to the DSDC problem and \( \omega_e(\xi, s) \) is the corresponding solution of the inverse problem:

\[
\frac{\partial \omega_e}{\partial s} = -\frac{\hat{c}}{(1 - b + s)^2} \frac{\partial^2 \omega_e}{\partial \xi^2} + \frac{\xi}{1 - b + s} \frac{\partial \omega_e}{\partial \xi} - \hat{c} \frac{\partial^4 \omega_e}{\partial \xi^4}
\]  

(12)

for

\[
0 \leq \xi \leq 1; \quad 0 < s \leq b
\]  

(13)

and

\[
\omega_e(\xi, 0) = \omega(\xi, b) \quad (i.c.),
\]  

(14)

\[
\omega_e(1, s) = 0 \quad (b.c.),
\]  

(15)

\[
(\omega_e)_{\xi\xi}(1, s) = 0 \quad (b.c.),
\]  

(16)

\[
(\omega_e)_{\xi}(0, s) = 0 \quad (b.c.).
\]  

(17)

Due to the increased order of the governing equation for the inverse problem, additional boundary conditions are needed. Condition (16) is proposed by Lions and reflects a zero curvature requirement.
2.1. Numerical aspects and results

The numerical solution of the above-mentioned inverse initial, boundary-value problem is obtained by the finite differences method. The following explicit algorithm has been used:

\[
\begin{align*}
\omega_{j+1}^n &= \omega_j^n + \frac{\Delta s}{\Delta \zeta(1-b+s_n)} \hat{\xi}_j(\omega_{j+1}^n - \omega_{j-1}^n) - \frac{\hat{\xi}}{\Delta \zeta^2(1-b+s_n)} (\omega_{j+1}^n - 2\omega_j^n + \omega_{j-1}^n) \\
&\quad - \frac{\varepsilon \Delta s}{\Delta \zeta^4} (\omega_{j+2}^n - 4\omega_{j+1}^n + 6\omega_j^n - 4\omega_{j-1}^n + \omega_{j-2}^n). 
\end{align*}
\]

The results of the ISDC problem are obtained for the following range of the regularization parameter: \(5 \times 10^{-3} < \varepsilon < 1.25 \times 10^{-2}\). For values of the regularization parameter outside this interval the solution of the inverse problem diverges significantly from the corresponding solution of the direct problem. We remark that the central subsidence \((\zeta = 0)\) approaches the solution of the direct problem for small values of the parameter \(\varepsilon\). However, as can be seen in Fig. 2, independent of \(\varepsilon\) and from the early steps of the numerical solution of the inverse problem, the curvature of the displacement line close to the right boundary diverges from the results of the direct problem. This observation will be elaborated below.

As is mentioned by Lions [3] the results of ISDC problem, using this kind of regularization, are depending on the depth variable: for large values of the depth variable \(s\), the solution of the ISDC problem diverges significantly from the data of the corresponding DSDC. Let \(H_d\) be the depth for which the direct problem has been solved. Due to the aforementioned divergence of results, the initial condition of the inverse problem could not be placed “earlier” than the value \(s \approx 0.25H_d\). The limited solution of the ISDC problem, using Lions’ regularization method, results from the strong diffusive character of the fourth-order regularization \(u_{\zeta \zeta \zeta \zeta}\)-term. This pathology is depicted in Figs. 2 and 3, where we show the comparison between the solutions of direct (solid line) and the regularized inverse problem (dotted line). Results are presented in the same form as Figs. 8–10, 12 and 13 of Part I.

In this point we must note that the explicit integration scheme (when it succeeds to yield a solution) gives the same results with those of an implicit scheme; the latter we have tried for the shake of comparison. Implicit integration schemes has been used for the solution of backwards heat conduction problems by Lion’s [3] and proved to have the same results as in this work.

![Fig. 2. Comparison of direct and inverse subsidence solution using Lion’s regularization.](image-url)
3. Mixed, $u_{\xi,s}$-regularization method

Above observations have prompted the use of a mixed regularization term that has derivatives in both $\xi$ and $s$. The choice that has been made is a $u_{\xi,s}$-regularization, considering appropriate initial and boundary conditions:

\[
\frac{\partial \omega_\xi}{\partial s} = -\frac{\tilde{c}}{(1-b+s)^2} \frac{\partial^2 \omega_\xi}{\partial \xi^2} - \frac{\zeta}{1-b+s} \frac{\partial \omega_\xi}{\partial \xi} + e \frac{\partial^4 \omega_\xi}{\partial \xi^2 \partial s^2}
\]

for

\[
0 \leq \zeta \leq 1; \quad 0 < s \leq b
\]

and

\[
\omega_\xi(\zeta, 0) = \omega(\zeta, b) \quad \text{(i.c.)},
\]

\[
(\omega_\xi)_s(\zeta, 0) = -\omega_s(\zeta, 0) \quad \text{(i.c.)},
\]

\[
\omega_\xi(1, s) = 0 \quad \text{(b.c.)},
\]

\[
(\omega_\xi)_s(0, s) = 0 \quad \text{(b.c.)}.
\]

It can be easily seen that the difference in the conditions between the two regularized ISDC problems refers to the replacement of the b.c., Eq. (16) by the i.c. Eq. (22). The additional initial condition can be derived through monitoring. It is possible, in general, to monitor not only the profile of surface subsidence distribution, Eq. (14) or (21), but also the downwards slope, Eq. (22) using for example some shallow depth reconnaissance technique, like shallow drilling.

3.1. Numerical aspects and results

The numerical solution of the above-mentioned initial, boundary-value problem, Eqs. (19)-(24), was obtained by using the method of finite differences. The algorithm that has been used is the following:
\[
\begin{align*}
&\left( -\frac{e}{\Delta \xi^2 \Delta s^2} \right) \omega_{j+1}^{p+1} + \left( \frac{1}{\Delta s} + \frac{2e}{\Delta \xi^2 \Delta s^2} \right) \omega_{j+1}^{p} + \left( -\frac{e}{\Delta \xi^2 \Delta s^2} \right) \omega_{j-1}^{p+1} \\
&\quad = + \left( -\frac{2e}{\Delta \xi^2 \Delta s^2} - \frac{\hat{c}}{(1 - b + s_n)^2 \Delta \xi^2} + \frac{\zeta(j)}{2(1 - b + s_n) \Delta \xi} \right) \omega_{j+1}^{p} \\
&\quad + \left( \frac{1}{\Delta s} + \frac{4e}{\Delta \xi^2 \Delta s^2} + \frac{2\hat{c}}{(1 - b + s_n)^2 \Delta \xi^2} \right) \omega_{j}^{p} \\
&\quad + \left( -\frac{2e}{\Delta \xi^2 \Delta s^2} - \frac{\hat{c}}{(1 - b + s_n)^2 \Delta \xi^2} - \frac{\zeta(j)}{2(1 - b + s_n) \Delta \xi} \right) \omega_{j-1}^{p+1} \\
&\quad + \left( + \frac{e}{\Delta \xi^2 \Delta s^2} \right) \omega_{j+1}^{p-1} + \left( -\frac{2e}{\Delta \xi^2 \Delta s^2} \right) \omega_{j-1}^{p-1} + \left( + \frac{e}{\Delta \xi^2 \Delta s^2} \right) \omega_{j-1}^{p-1}.
\end{align*}
\] (25)

As already mentioned, as “initial conditions” are used here results from the solution of the direct problem. Unlike the first regularization method, in the present one, the depth of the solution used as initial data, is more than twice the depth for which, the inverse problem with the previous regularization, was solved.

The results of the ISDC problem using \(u_{\xi_{\text{ss}}}-\)regularization are shown in Figs. 4 and 5, which are presented in the same form as Figs. 8–10, 12 and 13 of Part I. These results are obtained for the following values of the regularization parameter: \(10^{-2} < \varepsilon < 1\). For a choice of the regularization parameter greater than 1, the results diverge significantly from the direct solution. For values of \(\varepsilon\) lower than 0.01, the numerical results become unstable.

4. The von Neumann condition

The von Neumann condition insures the stability of the numerical solution, if [4]:

\[|\rho| \leq 1 + \kappa \Delta s\] (26)

with

\[0 \leq \kappa \leq M\] (27)

![Fig. 4. Comparison of direct and inverse subsidence solution using \(u_{\xi_{\text{ss}}}-\)regularization.](image)
for $\Delta \xi$ and $\Delta \varepsilon$ sufficiently small and with $|\rho|$ being the absolute value of the amplification factor. The right-hand inequality corresponds to the stability condition and the left-hand one is concerning the convergence to the exact solution.

4.1. The $u_{\xi\xi\varepsilon\varepsilon}$-regularization method

The von Neumann stability condition imposes as variables the increments $\Delta \xi$ and $\Delta \varepsilon$. Within a stability analysis, the variable $(1 - b + s)$ and the variable $\xi$ of the original problem are treated as parameters. Thus for given discretization and at the "time" step $n$ and for the "space" point $j$ we denote with $L_n$ the variable $(1 - b + s)$ and with $\xi_j$ the variable $\xi$. In Eq. (18) we set

$$\omega_j^n = A \rho^n e^{i \Delta \xi}.$$  

Following the formulation given above we get similarly:

$$|\rho| = |1 + \Delta \varepsilon f_n| \leq 1 + \Delta \varepsilon |f_n|,$$  

$$f_n = I \frac{\xi_j}{L_n \Delta \xi} \sin(\Delta \xi) - \frac{\sin^2(\Delta \xi)}{(\Delta \xi)^2} \left( \frac{4 \varepsilon - 16 \varepsilon}{L_n^2} \sin^2 \left( \frac{\Delta \xi}{2} \right) \right),$$  

where $I = \sqrt{-1}$ and

$L_n = 1 - b + s \in [0.18, 1.0]$;  

$\hat{c} = 0.1$;  

$\varepsilon \in [-\infty, +\infty]$;  

$\xi_j \in [0, 1.0]$.

For the left-hand side of the inequality (27) to be true we must ensure that:

$$\varepsilon < \frac{\hat{c}}{L_n^2} \frac{1}{c^2_{\Delta \xi}} - \frac{\sqrt{1 - \text{Im}^2(f_n)}}{c_{\Delta \xi}},$$  

Fig. 5. Comparison of direct and inverse subsidence solution using $u_{\xi\xi\varepsilon\varepsilon}$-regularization.
\[ c_{\Delta \xi} = 4 \frac{\sin^2 (\Delta \xi/2)}{\Delta \xi^2}, \]  
(36)

where \( \text{Im}(\cdot) \) means the imaginary part of a complex number. The measure of \( f_{\Delta \xi} \) is a decreasing function of \( \Delta \xi \) in the interval \((0, 1)\). Thus the lower upper bound of the measure of \( f_{\Delta \xi} \) is:

\[ \lim_{\Delta \xi \to 0} (|f_{\Delta \xi}|) = \sqrt{\frac{\xi_j^2 L_n^2 + \hat{c}^2 - \hat{c} \epsilon L_n^2 + \epsilon^2 L_n^4}{L_n^4}}. \]  
(37)

The previous quantity is an indicative amplification coefficient for the depth change of the \( \omega_\ell(\xi, s) \). If the previous expression is controlled, then the divergence between the numerical solution and the real solution can be also controlled.

Since we have ensured the stability of the problem, we are going to study the rate of convergence of the algorithm. It is obvious that the factor with which the rate of convergence becomes measurable is the measure of \( f_{\Delta \xi} \), which depends on \( \sin(\Delta \xi) \). So it can be proved [5] that

\[ |f_{\Delta \xi}| = O(\Delta \xi). \]  
(38)

### 4.2. The \( u_{\xi, \mu} \)-regularization method

By introducing Eq. (28) into Eq. (19) we obtain a quadratic equation for \( \rho \) as follows:

\[ (c_1 + b_1)\rho^2 - (2c_1 + b_1 + d_1)\rho + c_1 = 0, \]  
(39)

where

\[ c_1 = 2v(\Delta \xi)^2 L_n^2 c_\xi > 0, \]  
(40)

\[ b_1 = \Delta s(\Delta \xi)^4 L_n^2 > 0, \]  
(41)

\[ d_1 = (2\hat{c}(\Delta s)^2(\Delta \xi)^3 c_\xi) + (\sin(m \Delta \xi)\xi_j \Delta s^2(\Delta \xi)^3 L_n), \]  
(42)

\[ c_\xi = (1 - \cos(m \Delta \xi)) \]  
(43)

with

\[ L_n = 1 - b + s \in [0.18, 1.0], \]  
(44)

\[ \hat{c} = 0.1, \]  
(45)

\[ \epsilon \in [-\infty, +\infty], \]  
(46)

\[ \xi_j \in [0, 1.0], \]  
(47)

Let \( \rho_{1,2} \) be the complex roots of Eq. (39). According to the stability condition (Eqs. (26) and (27)), we are interested in the behavior of these roots as \( \Delta \xi \) and \( \Delta s \) tend to 0. For this limit, the roots of Eq. (39) tend to 1 and therefore \( |\rho_{1,2}| \) do the same. It can be proved [5] that the max \( |\rho_{1,2}| \) is bounded:

\[ |2 - b_c \Delta s + (b_c^2 + d_c)(\Delta s)^2 + O((\Delta s)^3)| \geq \max |\rho_{1,2}| \geq |1 + d_c(\Delta s)^2 + O((\Delta s)^3)|, \]  
(48)
where
\[
   b_e = \frac{1}{\varepsilon} > 0, \\
   d_e = \left( \frac{\hat{c}}{\varepsilon L_n^2} \right) + \text{I} \left( \frac{\xi_j}{\varepsilon L_n} \right)
\]
and \(\text{Re}(d_e) > 0, \text{Im}(d_e) > 0\), where \(\text{Re}(\cdot)\) denotes the real part of a complex number.

We notice that due to Eq. (48) the study of early convergence in terms of the norm of the amplification factor depends on the values of the parameters \(b_e\) and \(d_e\).

5. Truncation error of inverse problem—convergence

We are interested in the rate of convergence between the numerical and exact solutions as well as in the rate of convergence of the solutions individually. In previous paragraphs the rate of convergence was examined in terms of the values of the absolute value of the amplification factor. In this paragraph the rate of convergence will be examined in terms of the truncation error [4]. The truncation error for Lion’s \(u_{\xi_{44}}\)-regularization method is:
\[
   U_1(\tilde{x}) = \frac{\Delta s}{2} \left( \frac{\partial^2 \hat{\omega}_j}{\partial \xi^2} \right)_j^n - \frac{\Delta \xi^2}{6L_n} \left( \frac{\partial^3 \hat{\omega}_j}{\partial \xi^3} \right)_j^n - \frac{\hat{c}}{2L_n} \left( \frac{\partial^4 \hat{\omega}_j}{\partial \xi^4} \right)_j^n = O(\Delta s) + O(\Delta \xi^2).
\] (51)

On the other hand the truncation error for the \(u_{\xi_{44}}\)-regularization method is:
\[
   U_2(\tilde{x}) = \frac{\Delta s}{2} \left( \frac{\partial^2 \hat{\omega}_j}{\partial \xi^2} \right)_j^n + \frac{\xi_j \Delta \xi^2}{6L_n} \left( \frac{\partial^3 \hat{\omega}_j}{\partial \xi^3} \right)_j^n = O(\Delta s) + O(\Delta \xi^2),
\] (52)

where \(\tilde{x}_j^n\) corresponds to the discretized form of the exact solution.

According to the stability condition for the direct problem we must observe that \(0 < \Delta s \ll \Delta \xi\). Considering Eqs. (51) and (52) we can conclude that the main factor influencing the truncation error is the one that concerns \(\Delta \xi\). The point in space, which is influenced significantly by the numerical truncation error, is that close to the right boundary. At this space-point the third-derivative in \(\xi\) for both regularization and the fourth-derivative in \(\xi\) for \(u_{\xi_{44}}\)-regularization are important, since for small \(s\), the parameter \(L_n\) assumes small enough values. This can be verified in the graphs of the solution close to the right boundary, where there is significant difference between the solution of the inverse and direct problem. Notice that central subsidence \(\omega_x(0,s)\) is not affected by the truncation error.

6. Conclusions

In summary we can mention the following:

- The numerical solution of the ISDC concerning \(u_{\xi_{44}}\)-regularization can be derived for larger depths if compared with \(u_{\xi_{44}}\)-regularization. This is attributed to the strong diffusive character of the fourth-order derivative in \(\xi\).
• Stability in the sense of the von Neumann condition for both regularizations is ensured by right choices of the corresponding singular perturbation parameter $\varepsilon$ and the maximum $b$ of the problem that affects in turn the parameter $L_n$.

• The magnitude of the norm of the coefficient $f_A$ (Eq. (19)) as well as the coefficients $b_\varepsilon$ and $d_\varepsilon$ (Eqs. (38) and (39)) is not only affecting the stability of the algorithm but also the rate of convergence between numerical and exact solution, which is affected by the above-referred parameters.

• Convergence in terms of the truncation error is satisfactory, except on the right boundary and mostly due to the $\xi^\varepsilon$-regularization, where we have a significant difference from the solution of the direct problem.

Acknowledgements

The authors want to acknowledge the EU project Degradation and Instabilities in Geomaterials with Application to Hazard Mitigation (DIGA: HPRN-CT-2002-00220) in the framework of the Human Potential Program, Research Training Networks.

References