Numerical solution of KdV–KdV systems of Boussinesq equations
I. The numerical scheme and generalized solitary waves

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Abstract

Considered here is a Boussinesq system of equations from surface water wave theory. The particular system is one of a class of equations derived and analyzed in recent studies. After a brief review of theoretical aspects of this system, attention is turned to numerical methods for the approximation of its solutions with appropriate initial and boundary conditions. Because the system has a spatial structure somewhat like that of the Korteweg–de Vries equation, explicit schemes have unacceptable stability limitations. We instead implement a highly accurate, unconditionally stable scheme that features a Galerkin method with periodic splines to approximate the spatial structure and a two-stage Gauss–Legendre implicit Runge-Kutta method for the temporal discretization. After suitable testing of the numerical scheme, it is used to examine the travelling-wave solutions of the system. These are found to be generalized solitary waves, which are symmetric about their crest and which decay to small amplitude periodic structures as the spatial variable becomes large.

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1. Introduction

The coupled system

\begin{align*}
\eta_t + u_x + (\eta u)_x + \frac{1}{6}u_{xxx} &= 0, \\
\eta_t + u_x + u u_x + \frac{1}{6}u_{xxx} &= 0,
\end{align*}

(1)
of partial differential equations is one of a class of Boussinesq systems introduced in ref. [3]. We will refer to it as the ‘KdV–KdV system’ because of the special form of the dispersive terms (the terms with third-order derivatives). The variables in (1) are dimensionless, but unscaled. The system is an approximation to the full, two-dimensional Euler equations for surface wave propagation along a channel with a flat bottom filled with an ideal fluid when cross-channel variations can be ignored. The independent variable $x$ represents position along the channel, $t$ is proportional to elapsed

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time, \( \eta(x, t) \) is the deviation of the free surface from its rest position at the coordinate \( x \) at time \( t \) while \( u(x, t) \) is the horizontal velocity at the dimensionless height \( \sqrt{2/3} \) above the bottom (the undisturbed depth is 1 in the present variables), at the horizontal coordinate \( x \) at time \( t \). The initial-value problem for (1) with initial data \( \eta(x, 0) = \eta_0(x) \), \( u(x, 0) = u_0(x) \), \( x \in \mathbb{R} \), is linearly well-posed for \( (\eta, u) \in H^s \times H^s \) for \( s \geq 0 \) but is ill-posed in \( L^p \) for \( p \neq 2 \) (see [3]). For the nonlinear system, it was shown in ref. [4] that if \( (\eta_0, u_0) \in H^s \times H^s \) for \( s > 3/4 \), then there exist \( T > 0 \) and a unique solution of (1) such that \( (\eta, u) \in C(0, T; H^s)^2 \), \( (\eta, u) \in C(0, T; H^{s-3})^2 \). Moreover, it is readily seen that \( H^1 \)-solutions of (1) conserve the quantities \( I_1 = \int_{-\infty}^{\infty} u \, dx \), \( I_2 = \int_{-\infty}^{\infty} \eta \, dx \), \( I_3 = \int_{-\infty}^{\infty} u\eta \, dx \), and the Hamiltonian
\[
H = \int_{-\infty}^{\infty} (\frac{1}{6}\eta^3 + \frac{1}{6}u^3 - \frac{1}{2}u^2 - \frac{1}{2}x^3) \, dx.
\]

Related to (1) is its symmetric version, introduced in ref. [5], which has the form
\[
\eta_t + u_x + \frac{1}{2}(\eta u)_x + \frac{1}{6}u_{xxx} = 0,
\]
\[
u_t + \eta_x + \frac{1}{2}\eta u_x + \frac{1}{3}u u_x + \frac{1}{6}\eta_{xxx} = 0,
\]
and which reduces to a symmetric hyperbolic system when the dispersive terms are dropped. A local existence-uniqueness theory holds for the initial-value problem for (2) as well. Specifically, it is shown in ref. [5] that for \( (\eta_0, u_0) \in H^s \times H^s \), \( s > 3/2 \), there exists \( T > 0 \) and a unique solution of (2) such that \( (\eta, u) \in C(0, T; H^s)^2 \). Note that the symmetric system (2) conserves the \( L^2 \) norm, as the quantity \( \int_{-\infty}^{\infty} (\eta^2 + u^2) \, dx \) is invariant.

Another related system is a more general version of (1), namely
\[
\eta_t + u_x + (\eta u)_x + \frac{1}{6}u_{xxx} = 0,
\]
\[
u_t + \eta_x + uu_x + \left( \frac{1}{6} - \tau \right)\eta_{xxx} = 0,
\]
which contains a surface tension parameter, the Bond number \( \tau \), see, e.g. [8]. For \( \tau < 1/6 \) this system has a local existence-uniqueness theory similar to that of (1).

In this note, a numerical scheme for the periodic initial-value problem for (1)–(3) is developed. The semi-discretization obtained by discretizing the spatial structure via a standard Galerkin-finite element method with smooth splines on a uniform mesh yields a very stiff system of ordinary differential equations. Hence, it is not efficient to use explicit schemes for their temporal discretization. (The latter work well with the Bona-Smith and the ‘classical’ Boussinesq systems, both of which are not stiff in their semi-discrete rendition [1].) We resort instead to the two-stage implicit Runge-Kutta scheme of the Gauss–Legendre type, which has fourth order accuracy and possesses favorable nonlinear stability properties. The resulting nonlinear system of equations is linearized at each time step by Newton’s method coupled with an appropriate “inner” iterative scheme for solving the attendant linear systems efficiently, in the spirit of the analogous scheme for the scalar KdV equation in ref. [6]. The fully discrete scheme obtained in this way is unconditionally stable and highly accurate; its construction is outlined in Section 2 and the scheme is tested for accuracy in Section 3.

The numerical scheme just outlined is then used in an exploratory mode to investigate solutions of these particular Boussinesq models. As outlined in ref. [3], where a class of Boussinesq equations was derived, one of the criteria for accepting a particular member as an appropriate approximate model for solutions of the Euler equations is that the system in question have solitary-wave solutions like those of the Euler equations. Appealing to theory developed by Lombardi ([11] and the references therein), it is seen that (1) and (2) do not possess solitary waves decaying monotonically to zero at infinity with the exception of isolated, ‘embedded’ solitary waves. Instead, their travelling wave solutions are generalized solitary waves which are asymptotic to periodic solutions of small amplitude (ripples). Such generalized solitary waves have been shown to exist in gravity–capillary surface wave models and also for various other model equations and systems arising in water wave theory; see, e.g. [9–12,14,15]. We construct approximations to generalized solitary waves for (1)–(3) by solving numerically periodic boundary-value problems for the nonlinear systems of ordinary differential equations that travelling wave solutions of these systems satisfy. The resulting wave profiles are indeed of the above-described form and travel with constant speed and shape when inserted as initial values into the evolution code of Sections 2 and 3. The system (3) possesses embedded solitary waves of the usual strictly monotonic type for some which there exist exact solutions, as well as generalized solitary waves.

The present paper will be followed by a second one by the same authors, in which the evolution code is used to study the generation, interaction and stability of the generalized solitary wave solutions of (1) and (2).
2. The numerical method

Let \( r \geq 3 \) and consider the space \( S_h = S_h^r \) of periodic smooth splines of order \( r \) (degree \( r - 1 \)) on the interval \([a, b] \), on a uniform mesh with meshlength \( h = (b - a)/N \); hence \( dim S_h = N \). The standard Galerkin semidiscretization of the periodic initial-value problem for (1) on \([a, b] \) is a map \((\eta_h, u_h) : [0, T] \rightarrow S_h \times S_h \) satisfying

\[
\begin{align*}
(\eta_{ht}, \chi) &= -(u_{hx} + \eta_h u_x + \eta_h u_{hx}, \chi) + \frac{1}{6} (u_{hxx}, \chi), \quad \text{for all } \chi \in S_h, \\
(u_{ht}, \psi) &= -((\eta_h + u_h) u_x + \eta_h \psi_x) + \frac{1}{6} (\eta_{hx}, \psi_x), \quad \text{for all } \psi \in S_h,
\end{align*}
\]

(4)

and for which \( \eta_h(0) = \Pi_h \eta_0 \) and \( u_h(0) = \Pi_h u_0 \), where \( \Pi_h \) denotes any one of the projections such as interpolant, \( L^2 \)-projection, etc., such that for smooth, periodic \( v \) it is the case that \( \| \Pi_h v - v \| \leq ch^r \) for some constant \( c \) independent of \( h \). Here \( (\cdot, \cdot), \| \cdot \| \) denote, respectively, the \( L^2 \) inner product and norm on \([a, b] \). Define \( F_1, F_2 : S_h \times S_h \rightarrow S_h \) by requiring that

\[
\begin{align*}
(F_1(\eta, u), \chi) &= -(u_x + \eta u + u_x, \chi) + \frac{1}{6} (u_{xx}, \chi), \quad \text{for } \chi \in S_h, \\
(F_2(\eta, u), \psi) &= -u_x u_x + \eta \psi_x + \frac{1}{6} (\eta_{xx}, \psi_x), \quad \text{for } \psi \in S_h.
\end{align*}
\]

(5)

With this notation, the semidiscretization is a map \((\eta_h, u_h) : [0, T] \rightarrow S_h \times S_h \) satisfying

\[
\begin{align*}
\partial_t \eta_h &= F_1(\eta_h, u_h), \\
\partial_t u_h &= F_2(\eta_h, u_h),
\end{align*}
\]

(6)

for all \( t \in [0, T] \), and for which \( \eta_h(0) = \Pi_h \eta_0 \) and \( u_h(0) = \Pi_h u_0 \).

Consider the map \( Q : S_h \times S_h \rightarrow S_h \) defined for \( v, w \in S_h \) by \((Q(v, w), \chi) = (vw, \chi)\), for all \( \chi \in S_h \). We write \( Q(v) = Q(v, v) \). Let \( \Theta : S_h \rightarrow S_h \) be the linear operator defined for \( v \in S_h \) by \((\Theta v, \chi) = 1/6(v_{xx}, \chi') - (v_x, \chi)\) for all \( \chi \in S_h \). With this notation, the system (6) may be written in the form

\[
\begin{align*}
\partial_t \eta_h &= F_1(\eta_h, u_h) = Q(u_h, \eta_h) + \Theta u_h, \\
\partial_t u_h &= F_2(\eta_h, u_h) = \frac{1}{2} Q(u_h) + \Theta \eta_h.
\end{align*}
\]

(7)

The system (7) of ordinary differential equations is discretized in the temporal variable by the 2-stage Gauss–Legendre implicit Runge–Kutta method, which corresponds to the table

\[
\begin{array}{c|c|c|c|c|c}
   a_{11} & a_{12} & \tau_1 \\
   a_{21} & a_{22} & \tau_2 \\
   b_1 & b_2 &
\end{array}
= \begin{array}{c|c|c|c|c}
   \frac{1}{4} & \frac{1}{4} & -\frac{1}{2\sqrt{3}} & \frac{1}{2} & -\frac{1}{2\sqrt{3}} \\
   \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\
\end{array}
\]

The numerical scheme is now specified more precisely. Let \( i^n = nk, n = 0, 1, \ldots, J, \) where \( T = Jk \). We seek \( H^n, U^n \), by way of the intermediate stages \( H^{n,i}, U^{n,i} \) in \( S_h, i = 1, 2 \), which are the solutions of the \( 2 \times 2 \) system of nonlinear equations

\[
H^{n,i} = H^n + k \sum_{j=1}^{2} a_{ij} F_1(H^{n,j}, U^{n,j}), \quad U^{n,i} = U^n + k \sum_{j=1}^{2} a_{ij} F_2(H^{n,j}, U^{n,j}), \quad i = 1, 2,
\]

(8)

using the formulas

\[
H^{n+1} = H^n + \sum_{j=1}^{2} b_j F_1(H^{n,j}, U^{n,j}), \quad U^{n+1} = U^n + \sum_{j=1}^{2} b_j F_2(H^{n,j}, U^{n,j}).
\]

(9)

At each time step, the nonlinear system represented by (8) is approximately solved using Newton’s method as follows. Given \( n \geq 0 \), let \( H^{0,i}_0, U^{0,i}_0 \in S_h, i = 1, 2 \), be an accurate enough initial guess for \( H^{n,i}, U^{n,i} \), respectively. Then the iterates of Newton’s method for (8) (called the outer iterates) \( H^{n,i}_j, U^{n,i}_j, j = 1, 2, \ldots \), satisfy the linear system

\[
H^{n,i}_{j+1} = H^n - k \sum_{m=1}^{2} a_{im}(\Theta U^{n,m}_{j+1} + Q(H^{n,m}_{j+1}, U^{n,m}_{j+1}) + Q(H^{n,m}_{j+1}, U^{n,m}_{j+1})) = H^n - k \sum_{m=1}^{2} a_{im} Q(H^{n,m}_{j}, U^{n,m}_{j}), \quad i = 1, 2,
\]

(10)
\[
U_{j+1}^n - k \sum_{m=1}^{2} a_{im} (\Theta H_{j+1}^{n,m} + Q(U_{j+1}^{n,m}, U_{j+1}^{n,m})) = U^n - k \sum_{m=1}^{2} a_{im} \frac{1}{2} Q(U_j^{n,m}), \quad i = 1, 2.
\]

(11)

To solve efficiently the linear system represented by these equations, we add Eq. (10), \(i = 1\) to (11), \(i = 1\), and (10), \(i = 2\) to (11), \(i = 2\). We also subtract Eq. (11), \(i = 1\) from (10), \(i = 1\), and Eq. (11), \(i = 2\) from (10), \(i = 2\), producing four new equations which we write in operator form as a block diagonal \(2 \times 2\) linear system

\[
\begin{pmatrix}
A_1 & 0 \\
0 & A_2
\end{pmatrix}
\begin{pmatrix}
H_{j+1}^{n,1} + U_{j+1}^{n,1} \\
H_{j+1}^{n,2} + U_{j+1}^{n,2} \\
H_{j+1}^{n,2} - U_{j+1}^{n,2}
\end{pmatrix}
+ \begin{pmatrix}
b \\
b
\end{pmatrix}
= \begin{pmatrix}
r_1 \\
r_2 \\
q_1 \\
q_2
\end{pmatrix},
\]

where

\[
A_1 = \begin{pmatrix}
I - ka_{11}(\Theta \cdot + Q(\cdot, U_j^{n,1})) & -ka_{12}(\Theta \cdot + Q(\cdot, U_j^{n,2})) \\
-ka_{21}(\Theta \cdot + Q(\cdot, U_j^{n,1})) & I - ka_{22}(\Theta \cdot + Q(\cdot, U_j^{n,2}))
\end{pmatrix},
\]

\[
A_2 = \begin{pmatrix}
I - ka_{11}(-\Theta \cdot + Q(\cdot, U_j^{n,1})) & -ka_{12}(-\Theta \cdot + Q(\cdot, U_j^{n,2})) \\
-ka_{21}(-\Theta \cdot + Q(\cdot, U_j^{n,1})) & I - ka_{22}(-\Theta \cdot + Q(\cdot, U_j^{n,2}))
\end{pmatrix},
\]

\[
b = \begin{pmatrix}
-ka_{11} Q(H_{j+1}^{n,1}, U_{j+1}^{n,1}) & -ka_{12} Q(H_{j+1}^{n,2}, U_{j+1}^{n,2}) \\
-ka_{21} Q(H_{j+1}^{n,1}, U_{j+1}^{n,1}) & -ka_{22} Q(H_{j+1}^{n,2}, U_{j+1}^{n,2})
\end{pmatrix},
\]

\[
r_i = H^n + U^n - k \sum_{m=1}^{2} a_{im} \left( Q(H_j^{n,m}, U_j^{n,m}) + \frac{1}{2} Q(U_j^{n,m}) \right), \quad i = 1, 2,
\]

\[
q_i = H^n - U^n - k \sum_{m=1}^{2} a_{im} \left( Q(H_j^{n,m}, U_j^{n,m}) - \frac{1}{2} Q(U_j^{n,m}) \right), \quad i = 1, 2.
\]

The above system is split into two \(2 \times 2\) linear systems of equations, viz.

\[
\begin{pmatrix}
I - ka_{11} J_1(U_j^{n,1}) & -ka_{12} J_1(U_j^{n,2}) \\
-ka_{21} J_1(U_j^{n,1}) & I - ka_{22} J_1(U_j^{n,2})
\end{pmatrix}
\begin{pmatrix}
v_{j+1}^{n,1} \\
v_{j+1}^{n,2}
\end{pmatrix}
+ \begin{pmatrix}
b \\
b
\end{pmatrix}
= \begin{pmatrix}
r_1 \\
r_2
\end{pmatrix},
\]

(12)

and

\[
\begin{pmatrix}
I - ka_{11} J_2(U_j^{n,1}) & -ka_{12} J_2(U_j^{n,2}) \\
-ka_{21} J_2(U_j^{n,1}) & I - ka_{22} J_2(U_j^{n,2})
\end{pmatrix}
\begin{pmatrix}
w_{j+1}^{n,1} \\
w_{j+1}^{n,2}
\end{pmatrix}
+ \begin{pmatrix}
b \\
b
\end{pmatrix}
= \begin{pmatrix}
q_1 \\
q_2
\end{pmatrix},
\]

(13)

where \(J_1(\phi) = \Theta \phi + Q(\phi, \phi), J_2(\phi) = -\Theta \phi + Q(\phi, \phi), \text{ and } v_j^{n,i} = H_j^{n,i} + U_j^{n,i}, w_j^{n,i} = H_j^{n,i} - U_j^{n,i}, \text{ for } i = 1, 2.\)

Upon choosing a basis for \(S_h\), it becomes apparent that (12) and (13) represent two \(2N \times 2N\) linear systems for the coefficients of the Newton iterates. The following device was used to uncouple the two operator equations in each system. Evaluating all four entries of the matrices \(A_1, A_2\) at the point \(U^* \in S_h\) defined by \(U^* = (1/2)(U_0^{n,1} + U_0^{n,2})\) (which makes the operators in the entries of \(A_1\) and \(A_2\) independent of \(j\) and allows them to commute with each other), we may then write (12) and (13), respectively, as

\[
\begin{pmatrix}
I - ka_{11} J_1(U^*) & -ka_{12} J_1(U^*) \\
-ka_{21} J_1(U^*) & I - ka_{22} J_1(U^*)
\end{pmatrix}
\begin{pmatrix}
v_{j+1}^{n,1} \\
v_{j+1}^{n,2}
\end{pmatrix}
= \tilde{r} - \tilde{b},
\]

(14)

and

\[
\begin{pmatrix}
I - ka_{11} J_2(U^*) & -ka_{12} J_2(U^*) \\
-ka_{21} J_2(U^*) & I - ka_{22} J_2(U^*)
\end{pmatrix}
\begin{pmatrix}
w_{j+1}^{n,1} \\
w_{j+1}^{n,2}
\end{pmatrix}
= \tilde{q} - \tilde{b},
\]

(15)
where

\[
\tilde{r} = \begin{pmatrix} H^n + U^n \\ H^n + U^n \end{pmatrix} - k \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} Q(H_j^{n,1}, U_j^{n,1}) + \frac{1}{2} Q(U_j^{n,1}) \\ Q(H_j^{n,2}, U_j^{n,2}) + \frac{1}{2} Q(U_j^{n,2}) \end{pmatrix} \\
- k \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} J_1(U^*) - J_1(U_j^{n,1}) \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ J_1(U^*) - J_1(U_j^{n,2}) \end{pmatrix},
\]

and

\[
\tilde{q} = \begin{pmatrix} H^n - U^n \\ H^n - U^n \end{pmatrix} - k \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} Q(H_j^{n,1}, U_j^{n,1}) - \frac{1}{2} Q(U_j^{n,1}) \\ Q(H_j^{n,2}, U_j^{n,2}) - \frac{1}{2} Q(U_j^{n,2}) \end{pmatrix} \\
- k \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \begin{pmatrix} J_2(U^*) - J_2(U_j^{n,1}) \\ 0 \end{pmatrix} \begin{pmatrix} 0 \\ J_2(U^*) - J_2(U_j^{n,2}) \end{pmatrix},
\]

for \( j \geq 0 \). This form immediately suggests an iterative scheme for approximating \( v_{j+1}^{n,i} \) and \( w_{j+1}^{n,i} \), \( i = 1, 2 \). This scheme generates inner iterates denoted by \( v_{j+1}^{n,i,\ell} = H_{j+1}^{n,i,\ell} + U_{j+1}^{n,i,\ell} \) and \( w_{j+1}^{n,i,\ell} = H_{j+1}^{n,i,\ell} - U_{j+1}^{n,i,\ell} \) for given \( n, i, j \) and \( \ell = 0, 1, 2, \ldots \), (here \( v_{j+1}^{n,i,\ell} \) and \( w_{j+1}^{n,i,\ell} \) approximate \( v_{j+1}^{n,i} \) and \( w_{j+1}^{n,i} \), respectively) that are found recursively from the equations

\[
\begin{pmatrix} I - ka_{11}J_1(U^*) & -ka_{12}J_1(U^*) \\ -ka_{21}J_1(U^*) & I - ka_{22}J_1(U^*) \end{pmatrix} \begin{pmatrix} v_{j+1}^{n,1,\ell+1} \\ v_{j+1}^{n,2,\ell+1} \end{pmatrix} = \begin{pmatrix} v_{j+1}^{n,1,\ell} \\ v_{j+1}^{n,2,\ell} \end{pmatrix},
\]

and

\[
\begin{pmatrix} I - ka_{11}J_2(U^*) & -ka_{12}J_2(U^*) \\ -ka_{21}J_2(U^*) & I - ka_{22}J_2(U^*) \end{pmatrix} \begin{pmatrix} w_{j+1}^{n,1,\ell+1} \\ w_{j+1}^{n,2,\ell+1} \end{pmatrix} = \begin{pmatrix} w_{j+1}^{n,1,\ell} \\ w_{j+1}^{n,2,\ell} \end{pmatrix},
\]

for \( \ell \geq 1 \), where, for \( i = 1, 2 \),

\[
r_{j+1}^{n,i,\ell} = H^n + U^n - k \sum_{m=1}^{2} a_{1m} \left( Q(H_j^{n,i}, U_j^{n,i} - U_{j+1}^{n,i,\ell}) + \frac{1}{2} Q(U_j^{n,i}) + (J_1(U^*) - J_1(U_j^{n,i}))w_{j+1}^{n,i,\ell} \right),
\]

\[
q_{j+1}^{n,i,\ell} = H^n - U^n - k \sum_{m=1}^{2} a_{1m} \left( Q(H_j^{n,i}, U_j^{n,i} - U_{j+1}^{n,i,\ell}) - \frac{1}{2} Q(U_j^{n,i}) + (J_2(U^*) - J_2(U_j^{n,i}))w_{j+1}^{n,i,\ell} \right).
\]

The linear systems (16) and (17) can be solved efficiently as follows: Since \( a_{12}a_{21} < 0 \), it is possible, upon scaling the matrix on the left-hand sides by a diagonal similarity transformation, to write (16) and (17) as

\[
\begin{pmatrix} I - \frac{1}{4}kJ_1(U^*) & \frac{1}{4\sqrt{3}}kJ_1(U^*) \\ -\frac{1}{4\sqrt{3}}kJ_1(U^*) & I - \frac{1}{4}kJ_1(U^*) \end{pmatrix} \begin{pmatrix} v_{j+1}^{n,1,\ell+1} \\ \mu v_{j+1}^{n,2,\ell+1} \end{pmatrix} = \begin{pmatrix} r_{j+1}^{n,1,\ell} \\ \mu r_{j+1}^{n,2,\ell} \end{pmatrix},
\]

and

\[
\begin{pmatrix} I - \frac{1}{4}kJ_2(U^*) & \frac{1}{4\sqrt{3}}kJ_2(U^*) \\ -\frac{1}{4\sqrt{3}}kJ_2(U^*) & I - \frac{1}{4}kJ_2(U^*) \end{pmatrix} \begin{pmatrix} w_{j+1}^{n,1,\ell+1} \\ \mu w_{j+1}^{n,2,\ell+1} \end{pmatrix} = \begin{pmatrix} q_{j+1}^{n,1,\ell} \\ \mu q_{j+1}^{n,2,\ell} \end{pmatrix},
\]
where $\mu = 2 - \sqrt{3}$. These systems are equivalent to the two uncoupled complex $N \times N$ systems

$$(I - k\beta J_i(U^*))Z_i = R_i, \quad i = 1, 2,$$  

(20)

where $\beta = (1/4) + (1/4\sqrt{3})i$, and where $Z_i$ and $R_i$ are complex-valued functions with real and imaginary parts in $S_h$, depending upon $n$, $\ell$ and $j$, defined by $Z_1 = v_{n,1,\ell+1}^{j} + i\mu v_{n,2,\ell+1}^{j}$, $R_1 = r_{n,1,\ell}^{j} + i\mu r_{n,2,\ell}^{j}$ and $Z_2 = w_{n,1,\ell+1}^{j} + i\mu w_{n,2,\ell+1}^{j}$, $R_2 = q_{n,1,\ell}^{j} + i\mu q_{n,2,\ell}^{j}$.

In practice, only a finite number of outer and inner iterates are computed at each time step. Specifically, for $i = 1, 2$, $n \geq 0$, we compute approximations to the outer iterates $v_{n}^{j,i}, w_{n}^{j,i}$ for $j = 1, \ldots, J_{\text{out}}$, for some small positive integer $J_{\text{out}}$. For each $j$, $0 \leq j \leq J_{\text{out}} - 1$, $v_{n+1}^{j,i+1}$ and $w_{n+1}^{j,i+1}$ are approximated by the last inner iterates $v_{n}^{j,i,J_{\text{inn}}}, w_{n}^{j,i,J_{\text{inn}}}$ of the sequences of inner iterates $v_{n}^{j,i}, w_{n}^{j,i}$ that satisfy linear systems of the form (20). In practice, $J_{\text{inn}}$ and $J_{\text{out}}$ are such that

$$\left(\sum_{k=1}^{2} (\|U_{n+1}^{j,k,\ell} - U_{n}^{j,k,\ell}\|^2 + \|H_{n+1}^{j,k,\ell} - H_{n}^{j,k,\ell}\|^2)\right)^{1/2} \leq \varepsilon,$$

and

$$\left(\sum_{k=1}^{2} (\|U_{n}^{j,k,\ell} - U_{n-1}^{j,k,\ell}\|^2 + \|H_{n}^{j,k,\ell} - H_{n-1}^{j,k,\ell}\|^2)\right)^{1/2} \leq \varepsilon,$$

where $\|v\|_{\ell_2}$ denotes the Euclidean norm of the coefficients of $v \in S_h$ with respect to its basis, and $\varepsilon$ is usually taken to be $10^{-13}$ or $10^{-14}$.

Given $H_0^n, U^n$, the required starting values $H_0^{n-i}, U_0^{n-i}$ for the (outer) Newton iteration are computed by extrapolation from previous values as $H_0^{n-i} = \sum_{\mu=0}^{3} \alpha_{\mu,i} H_0^{n-\mu}$ and $U_0^{n-i} = \sum_{\mu=0}^{3} \beta_{\mu,i} U_0^{n-\mu}$ for $i = 1, 2$, where the coefficients $\alpha_{j,i}, \beta_{j,i}$ are such that $H_0^{n-1}$ and $U_0^{n-1}$ are the values at $t = n^{-i}$ of the Lagrange interpolating polynomial of degree at most 3 in $t$ that interpolates to the data $H_0^{n-j}$ and $U_0^{n-j}$ at the four points $t^{n-j}, 0 \leq j \leq 3$, respectively. (If $0 \leq n \leq 2$, we use the same linear combination, putting $U_0^0 = U_0$ and $H_0^t = H_0^t$ if $j < 0$. Here, $U_0^0 = \Pi_h u_0, H_0^t = \Pi_h \eta_0$.)

Analogous numerical schemes are readily derived for the coupled systems (2) and (3) as well.

3. Errors of the numerical method

Since a rigorous error analysis of the fully discrete scheme (8) and (9) approximating the coupled system (1) is not available, we performed an experimental investigation of its orders of convergence. The numerical solution was compared with the exact travelling wave solution (1) derived in ref. [7], valid for $x \in \mathbb{R}$ and given, for $\rho > 0$, by

$$\eta(x, t) = -\frac{1}{2} \left(1 + \frac{1}{6} \rho \right) + \frac{1}{4} \rho \text{sech}^2 \left(\frac{1}{2} \sqrt{\rho} (x - c_s t)\right),$$

$$u(x, t) = -\frac{\sqrt{2}}{2} \left(1 + \frac{1}{6} \rho \right) + c_s + \frac{1}{2\sqrt{2}} \rho \text{sech}^2 \left(\frac{1}{2} \sqrt{\rho} (x - c_s t)\right).$$

(21)

In (21) we took $c_s = 1$ and $\rho = 30$ and solved numerically the periodic initial-value problem for (1) on the spatial interval $[-5, 5]$ taking (21) for $t = 0$ as initial condition; both $\eta$ and $u$ differ from their constant, large $|x|$ asymptotic value by an amount of order $10^{-11}$ at the endpoints $x = \pm 5$ of the interval. Thus, treating the resulting truncated profiles as periodic introduced very small errors. The system was integrated using cubic and quintic splines on a uniform mesh with $h = 10/N$ up to $t = 1$ with time step $k = 1/M$. In these computations, in the case of cubic splines, we observed that the error tolerances mentioned in Section 2 were met if we took $J_{\text{out}} = 2$, and $J_{\text{inn}} = 4$ or 5 for the first and $J_{\text{inn}} = 1$ for the second outer iteration. (More outer and inner iterations were needed in the first three time steps due to lack of prior steps for the extrapolations in (26)). For quintic splines we observed that it was necessary to use $J_{\text{out}} = 2$ or 3 (mostly 2) coupled with $J_{\text{inn}} = 4$ or 5 for the first, $J_{\text{inn}} = 1$ to 3 for the second, and $J_{\text{inn}} = 1$ for the third outer iteration. We computed the discrete maximum error at $t = 1$ for $\eta$ as $\text{ME}_{\eta} = \max_{x} |H_0^M(x_1) - \eta(x_1, 1)|$ and the normalized $L^2$ error as $\text{LE}_{\eta} = \|H^0 - \eta(\cdot, t^n)\|/||\eta(\cdot, 0)||$ for $t^n = 1$, with analogous formulas for $u$. The $L^2$ norm is computed by Gauss quadrature with at least five nodes in every spatial subinterval.
Table 1
Spatial rates of convergence (cubic splines)

<table>
<thead>
<tr>
<th>$N$</th>
<th>$M$</th>
<th>$\text{ME}_u(t^n)$</th>
<th>Rate</th>
<th>$\text{LE}_u(t^n)$</th>
<th>Rate</th>
<th>$\text{ME}_u(t^n)$</th>
<th>Rate</th>
<th>$\text{LE}_u(t^n)$</th>
<th>Rate</th>
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<td>1.273e−7</td>
<td>4.13</td>
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Table 2
Spatial rates of convergence (quintic splines)

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<th>$\text{ME}_u(t^n)$</th>
<th>Rate</th>
<th>$\text{LE}_u(t^n)$</th>
<th>Rate</th>
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<th>Rate</th>
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<td>7.985e−11</td>
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</table>

To investigate the spatial order of convergence of the scheme, we took sufficiently small time steps (large enough values of $M$) and computed as usual experimental values of the rate of convergence as $\log(E_i/E_{i-1})/\log(h_i/h_{i-1})$, where $E_i$ was the error obtained with spatial meshlength $h_i$. The results, for cubic and quintic splines, are shown in Tables 1 and 2, respectively.

The tables confirm the expected theoretical rate of spatial convergence to be $r = 4$ for cubic and $r = 6$ for quintic splines. To investigate experimentally the temporal order of convergence, whose expected theoretical value is of course four, we computed with quintic splines taking $h = 10/N$ with values of $N$ shown in Table 3, and $k = h/2$. In this range of parameters, $h^6$ is about three orders of magnitude smaller than $k^4$ and we expect the temporal component of the error to be the dominant one. We approximated the temporal rate as $\log(E_i/E_{i-1})/\log(k_i/k_{i-1})$. The results of Table 3 yield approximately the expected theoretical value 4.

In the case of the symmetric KdV–KdV system (2), the conservation of the $L^2 \times L^2$ norm of the solution allows the rigorous derivation of optimal-order $L^2$ error estimates using the periodic spline quasi-interpolant as was done for the KdV equation in ref. [6]. The proof appears in ref. [13]. Numerical experiments confirm optimal-order errors of $O(k^4 + h^6)$ in this case as well.

Although the exact solution (21) is not a solitary wave, it may be used for testing the accuracy of numerical solutions of problems with solitary-wave type solutions. To this effect, we computed the numerical solution of the periodic initial-value problem for (1) using $h = 10^{-2}$, $k = 10^{-3}$, and (21) with $\rho = 30$, $c_i = 1$ on $[-5, 5]$ as before. In addition to the normalized $L^2$ error defined previously, we computed some other types of error indicators that are pertinent to the approximation of solitary waves (see [6]). Specifically, at $t = t^n$, we computed the (normalized) amplitude error $AE_\eta(t^n) = |(\eta_{\max} - H^n(x))/\eta_{\max}|$, where $\eta_{\max}$ is the maximum value of the $\eta$ profile (equal to 4.5 in our case) and $x^*$ is the point where $H^n$ achieves its maximum. This point is found by applying Newton’s method to the equation $d/dx(H^n(x)) = 0$ using a few iterations, and, as initial-value, the quadrature node where $H^n$ has a maximum. We
and up to the initial value problem for (1) was solved numerically with initial values the results of some of which will appear in a forthcoming paper. In the course of some early experiments, the periodic instability or due to an instability or blow-up of the solution of the system. As we shall see presently, this phenomenon also define an $L^2$ based (normalized) shape error as $\text{SE}_h(t^n) = \inf_\tau \| H^\tau - \eta(\cdot, \tau) \| / \| \eta(\cdot, 0) \|$, by first computing $\tau^\ast$ as the point near $t^n$ where $d/d\tau(\xi^2(\tau^\ast)) = 0$, with $\xi(\tau) = \| H^\tau - \eta(\cdot, \tau) \| / \| \eta(\cdot, 0) \|$, using Newton’s method with a few iterations and $t^0 = t^n - k$ as initial guess. We then set $\text{SE}_h(t^n) = \xi(\tau^\ast)$; the associated phase error is $\text{PE}_h(t^n) = \tau^\ast - t^n$.

We define the corresponding errors of $u$ similarly. Tables 4 and 5 show the evolution of these errors up to $t^n = 6$ for cubic and quintic splines, respectively.

It should be noted that for this problem, the numerical solution degenerates for larger values of $t$. For example, the $L^2$ errors increase with time and become $O(1)$ at about $t = 15.1$ for cubic splines and at about $t = 15.9$ for quintic. (The invariants $I_1$, $I_2$, $I_3$ and the Hamiltonian $H$ remain constant to 9 digits up to about $t = 17$ for cubic splines and up to $t = 18$ for quintic). This is a large amplitude problem and it is not clear whether this loss of accuracy of the numerical solution is because of accumulation of temporal error or consequence of some type of weak long-time instability or due to an instability or blow-up of the solution of the system. As we shall see presently, this phenomenon was not observed in simulations of small amplitude solutions; these remained accurate for very large time spans.

4. Generalized solitary waves

Using the numerical scheme described in the previous two sections, we performed many numerical experiments, the results of some of which will appear in a forthcoming paper. In the course of some early experiments, the periodic initial value problem for (1) was solved numerically with initial values $\eta(x, 0) = 0.3 \exp^{-(x+100)^2/25}$, $u(x, 0) = 0$ on $[-150, 150]$, using $h = 0.02$ (i.e. $N = 15000$) and $k = 0.004$. As expected, this initial profile resolved into two wave trains moving in opposite directions and led by solitary-like pulses. We tried to isolate a solitary wave (moving to the right) by iterative ‘cleaning’, cf. [2], i.e. by truncating the leading pulse, using it as new initial value, letting it propagate and distance itself from the trailing dispersive tail, truncate it again etc. After seven such iterations a ‘clean’, at least to the eye, solitary wave was produced, used as a new initial condition and allowed to evolve. At $t = 160$ the profile of the solution is shown in Fig. 1. In the magnified picture, we observe that small amplitude oscillations have been produced and accompany the main pulse. These oscillations do not appear to be an artifact of the numerical scheme; they prove to be invariant under changes in (small enough) values of $k$ and $h$, the choice of spline spaces and the time stepping method. A similar phenomenon was observed in the case of the symmetric system (2). In the forthcoming part II of this work we shall describe in detail these numerical experiments, the procedure of ‘cleaning’ that we used, and the observed properties of the small oscillations.

Such observations led us to ask whether these systems possess generalized solitary wave solutions, i.e. solitary wave pulses homoclinic to small amplitude oscillatory solutions. Such solutions are known to exist for the full Euler equations with small surface tension and other model nonlinear dispersive wave equations (cf., e.g. [9–12,14,15]). It turns out that the answer is affirmative since the vector field in $\mathbb{R}^4$ that defines the o.d.e. system corresponding to travelling wave solutions for (1) and (2) admits a $0^2 + i\omega$ resonance (see [11]).

Consider the system (1) and, following the notation and terminology of [11], seek travelling wave solutions of the form $\eta(x, t) = \eta(\xi)$, $u(x, t) = u(\xi)$, $\xi = x - ct$, and write $c_s = c + 1$. Substituting into (1), integrating once, setting the integration constants equal to zero and putting $u_1 = \eta$, $u_2 = \eta'$, $u_3 = u$, $u_4 = u'$ ($\eta = d/d\xi$), leads to the dynamical
Fig. 1. (a): Evolution of \( \eta \)-solitary wave of (1) produced (after 7 iterations) by iterative cleaning, \( t = 160 \). (b): Magnification of (a).

System

\[
\begin{align*}
    u'_1 &= u_2, \\
    u'_2 &= -6u_1 + 6(c + 1)u_3 - 3u_3^2, \\
    u'_3 &= u_4, \\
    u'_4 &= 6(c + 1)u_1 - 6u_3 - 6u_1u_3,
\end{align*}
\]

on \( \mathbb{R}^4 \). If \( U = (u_1, u_2, u_3, u_4)^T(\xi) \), then (22) may be written as \( U' = V(U, c) \equiv L(c)U + R(U) \), where

\[
L(c) = \begin{pmatrix}
0 & 1 & 0 & 0 \\
-6 & 0 & 6(c + 1) & 0 \\
0 & 0 & 0 & 1 \\
6(c + 1) & 0 & -6 & 0
\end{pmatrix}
\quad \text{and} \quad
R(U) = \begin{pmatrix}
0 \\
-3u_3^2 \\
0 \\
-6u_1u_3
\end{pmatrix}.
\]

It is easily seen that the spectrum of \( L(c) \), is the set \( \{ -\sqrt[6]{6\sqrt{-2} - c}, \sqrt[6]{6\sqrt{-2} - c}, -\sqrt[6]{6\sqrt{c}}, \sqrt[6]{6\sqrt{c}} \} \). In addition, the vector field \( V \) has the following properties:

(i) \( V(0, c) = 0 \) for all \( c \). \( U = 0 \) is a ‘fixed’ point of the system \( U' = V(U, c) \).
(ii) \( SV(U, c) = -V(SU, c) \), where \( S = \text{diag}\{1, -1, 1, -1\} \) (\( V \) is ‘reversible’).
(iii) The spectrum of \( L(0) \) is \( \{0, \pm i\omega\} \), \( \omega = 2\sqrt{3} \). The eigenvalue \( 0 \) is double and not semisimple. A corresponding basis for \( \mathbb{C}^4 \) is
\[
\phi_0 = (1, 0, 1, 0)^T, \quad \phi_1 = (0, 1, 0, 1)^T, \quad \phi_+ = \left( \frac{i}{2\sqrt{3}}, -1, -\frac{i}{2\sqrt{3}}, 1 \right)^T, \quad \phi_- = \left( -\frac{i}{2\sqrt{3}}, -1, \frac{i}{2\sqrt{3}}, 1 \right)^T,
\]
where \( \phi_0, \phi_\pm \) are eigenvectors corresponding to the eigenvalues \( 0, \pm i\omega \), respectively, and \( \phi_1 \) is a generalized eigenvector associated to the eigenvalue \( 0 \).

(iv) \( S\phi_0 = \phi_0 \).

(v) Denoting by \( \{\phi_0^*, \phi_1^*, \phi_\pm^*, \phi_\mp^*\} \) the corresponding dual basis (so that, e.g. \( \phi_1^* = (0, 1/2, 0, 1/2)^T \)), it transpires that \( c_{10} := \langle \phi_1^*, D_{cu}^2 V(0, 0)\phi_0 \rangle > 0 \) and \( c_{20} := 1/2\langle \phi_1^*, D_{uu}^2 V(0, 0)[\phi_0, \phi_0] \rangle \neq 0 \). (For (1) \( c_{10} = 6 \) and \( c_{20} = -9/2 \).)

By Theorem 7.1.1 of [11], it is inferred from these properties that there exist constants \( \sigma, \kappa_3, \kappa_2, \kappa_1, \kappa_0 > 0 \), such that, for \( c > 0 \) small enough, the vector field \( V(U, c) \) admits near \( U = 0 \):

(a) a one parameter family of periodic orbits \( p_{\kappa, c} \) of arbitrarily small amplitude \( \kappa \in [0, \kappa_3 c] \);
(b) for every \( \kappa \in [\kappa_1 c, \exp^{-\omega \pi/\sqrt{c_{10}^3}}] \), a pair of reversible (i.e. such that \( U(\xi) = SU(-\xi) \)) homoclinic connections to \( p_{\kappa, c} \) with one loop;
(c) no reversible homoclinic connections to \( p_{\kappa, c} \) with one loop if \( \kappa \in [0, \kappa_0 c, \exp^{-\pi \omega/\sqrt{c_{10}^3}}] \).

The term \textit{generalized solitary waves} is employed to indicate the profiles that are homoclinic to small amplitude periodic solutions. It is not hard to see that the symmetric system (2) also satisfies conditions (i)–(v) (with \( c_{20} = -6 \)). We conclude that both systems possess generalized solitary waves of small amplitude and speed \( c_s = 1 + c, \ c > 0 \)

![Image](a.png)

**Fig. 2.** Profile of \( \eta \)-generalized solitary wave for the system (1) with ripples with minimum amplitude \( \min \alpha = 0.000860, \ L = 15.2 \).
Fig. 3. Amplitude of the oscillations $\alpha$ vs. $L/\lambda_0$. The first ‘o’ corresponds to the solution plotted in Fig. 2. (System (1)).

Fig. 4. Comparison of generalized solitary waves of systems (1) (solid line) and (2) (dotted line) for $c = 0.2$. (a): Amplitude of ripples vs. $L/\lambda_0$. (b): Graph of $\eta$ for the generalized solitary waves with the minimum ripples.
small, which decay to exponentially small oscillatory solutions. Moreover, there is a critical size of the amplitude of the latter, below which there exist no generalized solitary waves.

To construct numerically such generalized solitary waves, solve the o.d.e. system (22) in the case of (1) (and the analogous system for (2)) imposing periodic boundary conditions on $u_i$ at the end points of the interval $[-L, L]$ for various values of $L$, for a given small positive value of $c$. Starting from an initial guess and using continuation, we employ as a solver the MATLAB® function bvpl4c, which implements a collocation method based on the 3-stage Lobatto IIIa quadrature rule. The mesh selection and the error control of the function are based on the residuals of the $C^1$ numerical solution that it provides.

In the case of (1), if we take $c = 0.2$ and as initial guess a sech$^2$-profile for all components $u_i$ and a relatively large value of $L$, we observe that the numerical method converges, with residuals of order $10^{-7}$ or smaller, to the $u$-profile shown in Fig. 2. (The $u$-profile has a similar form). In Fig. 2, the generalized solitary wave consists of a main ‘solitary’ pulse connected to a small amplitude periodic profile (ripples). The amplitude of the small oscillations varies with $L$ as Fig. 3 shows. In Fig. 3 we have plotted, for $c = 0.2$, the amplitude of the small oscillations versus $L/\lambda_0$, where $\lambda_0 = 2\pi/k$ is the wavelength corresponding to the wave number $k = \sqrt{6(c_s + 1)} = \sqrt{6(c + 2)}$ obtained from the dispersion relation for the linearized system (1). For $c = 0.2$ this gives $\lambda_0 \approx 1.72939$. (We have solved the o.d.e. system for $L = 15 + 0.05j$, $j = 0, \ldots, 100$, and computed the amplitude of the associated ripples.) The minimum amplitude occurs approximately at $L/\lambda_0 = (n/2) + (1/4)$, $n = 17, 18, \ldots$, and is constant to six decimal digits and equal to 0.000860. (Fig. 2 corresponds to $L = 15.2$, i.e. to the value where a minimum value $\min \alpha$ of $\alpha$ first occurs. Computing with different $L$’s corresponding to the minimum amplitude ripples – denoted by small circles in Fig. 3 – produces again Fig. 2 extended with ripples to the right and left.) The wavelength of the ripples in Fig. 2 is equal to about $\lambda \approx 1.73$, which is not far from the wavelength $\lambda_0$ predicted by the linearized problem. The maximum values of the amplitude occur near the values $L/\lambda_0 = (n + 1)/2$, $n = 17, 18, \ldots$; the o.d.e. code loses accuracy near these

![Fig. 5. Numerical integration in time of the generalized solitary wave with $c = 0.2$ for the system (2). The $u$-components of the solution is shown at $t = 0$ in (a), and at $t = 170$ in (b).](image-url)
The analogous o.d.e. system that corresponds to the symmetric system (2) is much easier to solve numerically as evidenced by the smaller residuals (of order $10^{-8}$ or better) that \texttt{bvp4c} returns. In this case, we were able to compute generalized solitary waves with $c = 0.2$ and $0.3$ and observed that the minimum amplitude of the ripples increases with $c$. When $c = 0.2$ we obtained $\min \alpha = 0.0012265$, while when $c = 0.3$, $\min \alpha = 0.0074614$. The maximum values also appear to increase.

In Fig. 4, we compare the amplitude of the ripples for $c = 0.2$ and the profile of the generalized solitary wave with minimum ripple amplitude for the two coupled KdV systems (1) and (2).

As a further test of the accuracy of the \texttt{bvp4c} function and the evolution code described in Sections 2 and 3, we took generalized solitary waves generated by solving the o.d.e. systems as initial data and let them evolve in time numerically using the evolution code. The results were quite satisfactory. Fig. 5 shows the graph of $\eta$ for the generalized solitary wave of the system (2) with $c = 0.2$ in $[-15, 15]$ at $t = 0$ and at $t = 170$. During this run (with $h = 0.1$, $k = 0.01$, $r = 4$, up to $T = 200$) the invariant quantity $\int_{-L}^{L} (\eta^2 + u^2) \, dx$ had the value $0.73237518000$ maintaining the eleven digits shown, while the quantities $\max \eta = 0.367190$, $\max u = 0.414506$, $c_s = 1.199999$ were conserved to six digits.

The analogous numerical integration in time for the system (1) was also quite accurate: With $h = 0.1$, $k = 0.01$, $r = 4$ and $T = 200$, the invariant quantities $\max \eta = 0.343972$, $\max u = 0.385334$, $c_s = 1.199999$, $I_1 = 0.73237518000$ were conserved to six digits.

Fig. 6. $\eta$-component of generalized solitary waves with two and three humps, $c = 0.2$, for the system (1).
Fig. 7. $\eta$-component of travelling wave solutions for system (3). (a) Embedded solitary wave, $\tau = 1/8$, $c_s = 3\sqrt{0.5}$, (b) Generalized solitary wave, $\tau = 0.01$, $c_s = 1.95$, (c) Travelling wave with damped oscillations, $\tau = 0.9$, $c_s = 2.9$.

1.5702595987, $I_2 = 1.4681669211$, $I_3 = 0.41614398659$, $H = -0.93347587389$ were conserved to the digits shown.

Let us also mention that using two or more sech$^2$-type, sufficiently separated pulses as initial guesses and integrating the o.d.e. system with bvp4c gave multi-humped generalized solitary waves as shown in Fig. 6 for the system (1). These multi-humped profiles evolved as travelling wave solutions to high accuracy when used as initial values in the evolution code. (Similar results obtain for the system (2)). We note again that the system (3) has exact solitary waves at least for $(1/12) < \tau < (1/6)$. For this range of Bond number, it has been found, see [8], that (3) has exact solitary waves of the form $\eta(\xi) = \eta^0 \text{sech}^2(\lambda \xi), u(\xi) = B\eta(\xi)$, where $\xi = x + x_0 - c_s t$, $\eta^0 = (3 - 36\tau)/(-2 + 12\tau), \lambda = \sqrt{\eta^0}, B = \sqrt{2 - 12\tau}, c_s = (2 - B^2)/B$. For other values of $\tau$ and $c_s$ we found numerically (by solving the associated o.d.e.
system with periodic boundary conditions using bvp4c, other types of travelling wave solutions shown in Fig. 7, cf. [13].

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