ON SOME UNCONDITIONALLY STABLE, HIGHER ORDER METHODS FOR THE NUMERICAL SOLUTION OF THE STRUCTURAL DYNAMICS EQUATIONS†

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SUMMARY
Third- and fourth-order accurate Nørsett rational approximations to the exponential and associated semi-implicit Runge-Kutta methods are used for the construction of efficient, accurate and unconditionally stable schemes for the direct numerical integration of the linear, nonhomogeneous, second-order equations of structural dynamics.

INTRODUCTION
In this note we consider the linear, second-order system of ordinary differential equations

\[ M\ddot{y} + C\dot{y} + Ky = f(t) \quad t > 0 \]
\[ y(0) = y_0 \quad \dot{y}(0) = \dot{y}_0 \]

arising, for example, in the dynamic analysis of structures as well as in various semi-discretizations of the damped wave equation. We shall assume that \( M, C, K \) are sparse, symmetric, positive definite matrices and write (1) in its usual first-order formulation

\[
\begin{bmatrix}
I & 0 \\
0 & M
\end{bmatrix} \frac{d}{dt} \begin{bmatrix}
y \\
\dot{y}
\end{bmatrix} + \begin{bmatrix}
0 & -T \\
K & C
\end{bmatrix} \begin{bmatrix}
y \\
\dot{y}
\end{bmatrix} = \begin{bmatrix}
0 \\
f(t)
\end{bmatrix} \quad t > 0
\]

\[ \begin{bmatrix}
y(0) \\
\dot{y}(0)
\end{bmatrix}^T = [y_0, \dot{y}_0]^T \]

or, more compactly, as

\[
\dot{\mathcal{Y}} = \mathcal{A}\mathcal{Y} + \mathcal{G} \quad \mathcal{Y}(0) = \mathcal{Y}_0
\]

\[
\mathcal{A} = \begin{bmatrix}
0 & I \\
-M^{-1}K & -M^{-1}C
\end{bmatrix}
\]

and \( \mathcal{Y}(t) = [y(t), \dot{y}(t)]^T, \mathcal{Y}_0 = [y_0, \dot{y}_0]^T, \mathcal{G}(t) = [0, M^{-1}f(t)]^T. \)

Recently, Brusa and Nigro derived a (globally) third-order accurate, unconditionally stable single-step method for the direct numerical integration of (2) which may be efficiently implemented taking into account the sparsity of \( M, C, K. \) In this note we wish to observe that in the homogeneous case \((f = 0)\) this method may be derived directly by means of a well-known rational approximation to the exponential function, a number of the class of the Nørsett approximations; we then review briefly the stability and accuracy properties of these methods.

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and specifically point out an effective implementation of these schemes in the case of systems like (2). Finally, establishing that they may be viewed as semi-implicit Runge–Kutta schemes (cf. the works of Nørsett, Crouzeix and Alexander among others), we point out efficient implementations of some third- and fourth-order accurate methods of this class in the non-homogeneous case \( f(t) \neq 0 \). We note that semi-implicit Runge–Kutta methods for systems like (2) have been considered by Crouzeix in an abstract setting and that Smith has considered the application (and the comparison) of some of such methods in the case of (2) with \( f \) independent of \( t \) and has reported on numerical experiments.

**Nørsett Rational Approximations to the Exponential**

In this section we consider only the homogeneous case \( f = 0 \) in (2)). Let \( h > 0 \) be a constant time step. Then, the solution of (3) satisfies

\[
\mathcal{Y}(t + h) = e^{zh}\mathcal{Y}(t) \quad t \geq 0
\]  

(4)

Let \( r(z) \) denote a rational approximation to the (complex) exponential function \( e^z \). It is well known that, in view of (4), one may define a scheme for the approximation \( \mathcal{Y}^n \) of the solution \( \mathcal{Y}(t^n) \) of (3) at the time level \( t^n = nh, n = 0, 1, 2, \ldots \), using the recursion formula

\[
\mathcal{Y}^{n+1} = r(\mathcal{A}h)\mathcal{Y}^n \quad n \geq 0
\]  

(5)

where \( \mathcal{Y}^0 = [\mathcal{Y}_0, \mathcal{Y}_0]^T \).

A particular class of rational approximations \( r(z) \), systematically introduced and studied by Nørsett, turns out to be particularly effective for the numerical solution of first-order homogeneous problems of the general form (3) and also for the particular case of the system (2). For each \( \nu = 1, 2, 3, \ldots \), \( r(z) \) is given as a one-parameter family (for \( \beta \) positive) by

\[
r_{\nu}(\beta, z) = (-1)^{\nu}\left[ \sum_{k=0}^{\nu} L^{(v-k)}(1/\beta)(\beta z)^k \right]/(1 - \beta z)^\nu
\]  

(6)

where \( L^{(j)} \) denotes the \( j \)th derivative of the Laguerre polynomial \( L_{\nu} \) of degree \( \nu \) given by

\[
L_{\nu}(z) = \sum_{j=0}^{\nu} \frac{(-1)^j}{j!} \begin{pmatrix} \nu \\ j \end{pmatrix} z^j
\]

The rational function (6) may also be written as

\[
r_{\nu}(\beta, z) = 1 - \sum_{j=0}^{\nu-1} P_{j}(\beta)[-(1 - \beta z)]^{j+1}
\]  

(7)

where

\[
P_{j}(\beta) = \frac{-\beta^j}{j + 1} \frac{d}{dz} L_{j+1}(1/\beta)
\]

As a consequence of the fact that the denominator of \( r_{\nu}(\beta, z) \) is a power of \( 1 - \beta z \), there follows that (5) may be implemented as follows. For each \( n \)

- define \( \mathcal{Y}_0 = \mathcal{Y}_n \)
- compute recursively \( (1 - \beta h\mathcal{A})\mathcal{W}_{j+1} = h\mathcal{A}\mathcal{W}_j \quad j = 0, 1, \ldots, \nu - 1 \)
- set \( \mathcal{Y}_{n+1} = \mathcal{Y}_n - \sum_{j=0}^{\nu-1} P_{j}(\beta)[(1 - \beta z)]^{j+1}\mathcal{W}_{j+1} \)
Substituting in the above the $2 \times 2$ block matrix $\mathcal{A}$ of (3), we obtain the following approximation scheme for the system (2) in the homogeneous case. For each $n$

- define $W_0 = \Psi_0$

- compute recursively $W_j = [u_j, v_j]^T$, $j = 0, 1, \ldots, \nu - 1$ by

\[
(M + \beta h C + \beta^2 h^2 K)v_{j+1} = h K(u_j + \beta h v_j) + h C v_j
\]

\[
u_{j+1} = \beta h v_{j+1} - h v_j
\]

(8)

set $\Psi_{n+1} = \Psi_n - \sum_{j=0}^{\nu-1} P_i(\beta)(-1)^{i+1} W_{j+1}$

Note that due to the special form of $\mathcal{A}$, the computation of the components $u_j, v_j$ of the intermediate vectors $W_j$ is decoupled. Thus, if the Cholesky $L L^T$ decomposition of the symmetric, positive definite matrix $M + \beta h C + \beta^2 h^2 K$ (which has the same size and sparse structure as $M, C, K$) is performed only once and the factor $L$ is stored, then the algorithm (8) needs, for each $n, \nu$ back-substitutions with the factors $L, L^T$ and $2\nu$ matrix-vector multiplications with matrices of the same size as $M, C, K$ for the formation of right-hand sides.

The choice of the parameters $\nu, \beta$ in (6) leads to specific properties of the rational function $r_{\nu}(\beta, z)$ as an approximation to $e^z$, which in turn lead to corresponding stability and accuracy properties of the method (8). Such properties have been extensively studied and documented by Nørsett and others. Generally speaking, for each $\nu, r_{\nu}(\beta, z) = e^z + O(z^{\nu+1})$, a fact which implies that the method (8) (provided it is stable) has (global) order of accuracy $\nu$. For each $\nu$ there are exceptional values of $\beta = \tilde{\beta}$ (the inverses of the roots of $d/dz L_{\nu+1}(z)$) for which the order of accuracy is increased to $\nu + 1$, the maximal order of accuracy for such methods; we shall refer to the corresponding methods as the optimal order ones.

As a consequence of our hypotheses on the matrices $M, C, K$ it follows that the eigenvalues of $\mathcal{A}$ lie in the left half of the complex $z$-plane. As a consequence A-stable methods are needed for the unconditional stability of (8). Recall that A-stability is equivalent to requiring that $|r(z)| \leq 1$ for all $z$ with $\text{Re } z \leq 0$. (A method is called L-stable if it is A-stable and the associated $r(z)$ tends to zero as $|z| \to \infty$ with $\text{Re } z < 0$.) It is known that there exist only four optimal order A-stable Nørsett methods with $\nu = 1, 2, 3, 5$, respectively. The corresponding (global) orders of accuracy for (8) are then 2, 3, 4, 6.

For the convenience of the reader and for further reference we review now a few specific facts about Nørsett methods with $\nu = 2$ and 3 of practical interest. For the implementation of these methods we use, in (8), $P_0(\beta) = 1$, $P_1(\beta) = \beta - 1/2$, $P_2(\beta) = \beta^2 - \beta + 1/6$.

(i) $\nu = 2$

The value $\beta = \hat{\beta}_2 = (1 + \sqrt{3}/3)/2$ leads to the only optimal order method for $\nu = 2$ (i.e. having order of accuracy 3) which is A-stable. The method (which is not L-stable) is also known as the Calahan method. It may be shown, by a suitable change of variables, that (8) with $\nu = 2, \beta = \hat{\beta}_2$ is equivalent to the method proposed by Brusa and Nigro in the homogeneous case.

The values $\hat{\beta}_{1,2} = 1 \pm (\sqrt{2}/2)$ (inverses of the roots of $L_2(z)$) give two L-stable methods of order of accuracy 2 for which (cf. (6)) the numerator of $r_2(\beta, z)$ is a linear polynomial. This fact does not seem to lower the number of matrix-vector operations in the case of the second-order problem (1) written as a first-order system.
(ii) $\nu = 3$

The only $A$-stable optimal (i.e. of order of accuracy 4) method corresponds to the value

$$\beta_3 = \frac{1}{\sqrt{3}} \cos \frac{\pi}{18} + \frac{1}{2} \equiv 1.06856790$$

(9)

This method is not $L$-stable. For $\phi = \arctan \sqrt{2}/4$, the method corresponding to

$$\beta_3 = \sqrt{2} \cos \left( \frac{\phi + 4\pi}{3} \right) + 1 \equiv 0.43586652$$

(10)

is of order of accuracy 3 and $L$-stable; the corresponding $r_3(\beta, z)$ has quadratic numerator, since $\beta_3$ is the inverse of one of the roots of $L_3(z)$ (the other two roots do not lead to $A$-stable methods).

Note that other rational approximations to the exponential may be used in (5) instead of (6). For example, recently, one of us\textsuperscript{12} showed how to implement efficiently the $(2,2)$-Padé approximation to $e^z$ (using complex arithmetic) in the context of the generally nonhomogeneous system (2).

## DIAGONALLY IMPLICIT RUNGE-KUTTA METHODS

In the case of the non-homogeneous ($f \neq 0$) system (2) we shall restrict ourselves to some semi-implicit Runge–Kutta (RK) methods\textsuperscript{3,5} that reduce for $f = 0$ to particular examples of third- and fourth-order accurate Nørsett schemes given in the previous section. We first introduce some notation. A $\nu$-stage (implicit) RK method for the system of differential equations

$$\dot{y}(t) = f(t, y(t))$$

may be determined by a table of $\nu^2 + 2\nu$ constants of the form

$$\begin{array}{cccc|c}
  a_{11} & \ldots & a_{1\nu} & \tau_1 \\
  \vdots & & \vdots & \vdots \\
  a_{\nu 1} & \ldots & a_{\nu \nu} & \tau_\nu \\
  \hline
  b_1 & \ldots & b_\nu
\end{array}$$

(11)

The corresponding algorithm for the computation of the approximations $y_n$ to $y(t^n) = y(nh)$ is then given by

$$k_{n,i} = f \left( t_{n,i}, y_n + h \sum_{j=1}^{\nu} a_{ij} k_{n,j} \right) \quad i = 1, 2, \ldots, \nu$$

$$t_{n,i} = t_n + \tau_i h$$

(11a)

$$y_{n+1} = y_n + h \sum_{j=1}^{\nu} b_j k_{n,j}$$

The method (11) is called semi-implicit if $a_{jj} = 0$ for $j > i$ and diagonally implicit\textsuperscript{5} if it is semi-implicit and the diagonal elements $a_{ii}$ are equal. The methods of the latter class may be implemented efficiently for linear and nonlinear systems of differential equations and reduce in the case $\dot{y} = y$ to the Nørsett approximations to the exponential with denominator $(1 - a_{ii} z)^\nu$. In the sequel, a $\nu$-stage RK method of order of accuracy $p$ will be referred to as a $(\nu, p)$ method.
Consider now applying a \( \nu \)-stage diagonally implicit RK method (of the form (11) with \( a_{ii} = 0 \) \( i < j, \ a_{ii} = \beta \)) to the system (3). Letting \( \mathcal{G}_{n,i} = \mathcal{G}(t_{n,i}) = \mathcal{G}(t_n + \tau h) \), we obtain

\[
(\mathcal{I} - \beta h M) \mathcal{K}_{n,i} = \mathcal{A} \mathcal{Y}_n + \mathcal{G}_{n,i}
\]

\[
(\mathcal{I} - \beta h M) \mathcal{K}_{n,i} = \mathcal{A} \left( \mathcal{Y}_n + h \sum_{i=1}^{i-1} a_{ii} \mathcal{K}_{n,i} \right) + \mathcal{G}_{n,i} \quad 2 \leq i \leq \nu
\]

\[
\mathcal{Y}_{n+1} = \mathcal{Y}_n + h \sum_{i=1}^{\nu} b_i \mathcal{K}_{n,i}
\]

which may be implemented, in the case of the system (2), as follows. Let \( \mathcal{Y}_n = [u_n, v_n]^T \), \( \mathcal{K}_{n,i} = [u_{n,i}, v_{n,i}]^T \), \( f_{n,i} = f(t_n + \tau h) \) and obtain

\[
\begin{align*}
\mathbf{r}_1 &= u_n \\
\mathbf{s}_1 &= v_n \\
(M + \beta h C + \beta^2 h^2 K) \mathbf{v}_{n,i} &= -K(\mathbf{r}_i + \beta h \mathbf{s}_i) - C \mathbf{s}_i + f_{n,i} \\
u_{n,i} &= \beta h v_{n,i} + \mathbf{s}_i \\
\mathbf{r}_i &= \mathbf{r}_1 + h \sum_{i=1}^{i-1} a_{ii} \mathbf{u}_{n,i} \\
\mathbf{s}_i &= \mathbf{s}_1 + h \sum_{i=1}^{i-1} a_{ii} \mathbf{v}_{n,i} \\
\mathbf{r}_{n+1} &= \mathbf{r}_n + h \sum_{i=1}^{\nu} b_i \mathcal{K}_{n,i}
\end{align*}
\]

(12)

Hence, the \( \nu \)-stage method (12) requires for each \( n \), \( \nu \) back-substitutions with the Cholesky decomposition factors \( L, L^T \) of the matrix \( M + \beta h C + \beta^2 h^2 K \), \( 2 \nu \) matrix-vector multiplications for the formation of right-hand sides and \( \nu \) evaluations of the function \( f(t) \) at the points \( t_n + \tau h, 1 \leq i \leq \nu \).

We now consider specific examples\(^4,5\) of such methods. For \( \nu = 2 \) the (2, 3) method

\[
\begin{array}{c|cc}
\beta_2 & 0 & \beta_2 \\
1 - 2\beta_2 & \beta_2 & 1 - \beta_2 \\
\hline
1/2 & 1/2
\end{array}
\]

(13)

reduces to the \( A \)-stable Calahan scheme in the homogeneous case. For \( \nu = 3 \), the table

\[
\begin{array}{c|ccc}
\beta_3 & 0 & 0 & \beta_3 \\
c & \beta_3 & 0 & c + \beta_3 \\
b_1 & b_2 & \beta_3 & 1 \\
\hline
\hline
b_1 & b_2 & \beta_3 & 1 \\
1/2 - \beta_3 & \beta_3 & 0 & \frac{1}{2} \\
2\beta_3 & 1 - 4\beta_3 & \beta_3 & 1 - \beta_3 \\
b_1 & b_2 & b_3
\end{array}
\]

(14)

gives a (3, 3) method, an extension of the \( L \)-stable Nørsett method with \( \nu = 3 \), \( \beta = \beta_3 \). Finally, for \( \nu = 3 \), the table

\[
\begin{array}{c|ccc}
\beta_3 & 0 & 0 & \beta_3 \\
\frac{1}{2} - \beta_3 & \beta_3 & 0 & \frac{1}{2} \\
2\beta_3 & 1 - 4\beta_3 & \beta_3 & 1 - \beta_3 \\
b_1 & b_2 & b_3
\end{array}
\]

(15)

gives a (3, 3) method, an extension of the \( L \)-stable Nørsett method with \( \nu = 3 \), \( \beta = \beta_3 \). Finally, for \( \nu = 3 \), the table
gives a (3, 4) method which may be viewed as an extension of the $A$-stable optimal order Nørsett scheme with $\nu = 3$.

It may be noted that extensions of the Nørsett methods to the nonhomogeneous case may be obtained by many other ways. For example, using the formula for the explicit solution of the nonhomogeneous system (3), expanding the integrand in a Taylor series about $t^n$, using the rational approximation to $e^{\omega t}$, integrating and replacing the time derivatives of $f$ by appropriate difference quotients, one may obtain formulae for the nonhomogeneous equations corresponding to any given single-step method of the form (5) for the homogeneous problem. This has been done for example for the Calahan scheme; the resulting formula requires three function evaluations per time step, whereas the implementation (13) of the method as a (2, 3) RK formula requires two.

NUMERICAL RESULTS

We first present some numerical results using the methods (13), (14) and (15) on a model problem. (See also the cited works for other reports of numerical experiments with (13), (14). We consider a problem arising in the Galerkin semi-discretization (using piecewise linear, continuous functions in space) of a damped one-dimensional wave equation with zero boundary conditions on the interval $[0, 1]$. With $\Delta x = 1/(N+1)$, $N$ integer, $M$, $C$, $K$ are symmetric, positive definite tridiagonal matrices with elements $M_{ii} = 2\Delta x/3$, $M_{i,i-1} = M_{i,i+1} = \Delta x/6$, $C_{ii} = 2\Delta x(1+i\Delta x)/3$, $C_{i,i-1} = C_{i,i+1} = (\Delta x/6) + (2i+1)\Delta x^2/12$, $K_{ii} = 2/\Delta x$, $K_{i,i-1} = K_{i,i+1} = -1/\Delta x$; the initial conditions and the right-hand side are chosen so that the solution vector (for the interior nodes) is $y_i(t) = i\Delta x \left[ \frac{1}{1-i\Delta x} \right] \sin \pi t$, $1 \leq i \leq N$. We took $N = 9$. All computations were done in double precision on the IBM 370/3031 of the University of Tennessee, Knoxville. Table I shows the maximum absolute error at $t = 1$ for the three methods with varying time steps. In Table II we tabulate the number of time steps that each method required to achieve a maximum absolute error at $t = 1$ of $10^{-4}$, $10^{-6}$ and $10^{-7}$.

Table I. Maximum error at $t = 1$

<table>
<thead>
<tr>
<th>No. steps</th>
<th>Method</th>
<th>Eqn (13) (2, 3) RK</th>
<th>Eqn (14) (3, 3) RK</th>
<th>Eqn (15) (3, 4) RK</th>
</tr>
</thead>
<tbody>
<tr>
<td>20</td>
<td>0.859E-4</td>
<td>0.216E-4</td>
<td>0.321E-4</td>
<td></td>
</tr>
<tr>
<td>25</td>
<td>0.419E-4</td>
<td>0.107E-4</td>
<td>0.147E-4</td>
<td></td>
</tr>
<tr>
<td>40</td>
<td>0.936E-5</td>
<td>0.246E-5</td>
<td>0.260E-5</td>
<td></td>
</tr>
<tr>
<td>50</td>
<td>0.463E-5</td>
<td>0.123E-5</td>
<td>0.112E-5</td>
<td></td>
</tr>
<tr>
<td>60</td>
<td>0.261E-5</td>
<td>0.702E-6</td>
<td>0.555E-6</td>
<td></td>
</tr>
<tr>
<td>80</td>
<td>0.107E-5</td>
<td>0.290E-6</td>
<td>0.182E-6</td>
<td></td>
</tr>
<tr>
<td>100</td>
<td>0.536E-6</td>
<td>0.147E-6</td>
<td>0.761E-7</td>
<td></td>
</tr>
</tbody>
</table>

Table II. Number of steps to achieve error levels $10^{-4}$, $10^{-6}$ and $10^{-7}$

<table>
<thead>
<tr>
<th>Error level</th>
<th>Method</th>
<th>Eqn (13) (2, 3) RK</th>
<th>Eqn (14) (3, 3) RK</th>
<th>Eqn (15) (3, 4) RK</th>
</tr>
</thead>
<tbody>
<tr>
<td>$10^{-4}$</td>
<td>40</td>
<td>26</td>
<td>28</td>
<td></td>
</tr>
<tr>
<td>$10^{-6}$</td>
<td>82</td>
<td>54</td>
<td>52</td>
<td></td>
</tr>
<tr>
<td>$10^{-7}$</td>
<td>173</td>
<td>114</td>
<td>94</td>
<td></td>
</tr>
</tbody>
</table>
Since the number of operations per time step for the methods (14) and (15) is approximately 3/2 times the number of operations of the (2, 3) RK method (13), we may infer from Table II that the cost of achieving any one of the three levels of error for this problem using methods (13) and (14) is about the same; the three-stage method (14) is slightly more economical. At the error level $10^{-4}$ the third-order methods (13), (14) cost less than the fourth-order method (15), which, on the other hand, is slightly more economical than the third-order methods at the error level $10^{-6}$ and significantly less expensive at the error level $10^{-7}$.

We have also compared the three methods with respect to the numerical dispersion (phase error) and numerical damping (dissipation) that they introduce in approximating the model scalar problem

$$\ddot{y} + \omega^2 y = 0, \quad \omega > 0$$

(16)

For similar comparisons see the cited works\(^1,6\) and also Park.\(^{13}\) Let $r(i\omega h)$ be the approximation to the exponential $e^{i\omega h}$ using the rational function $r(z)$. For $x = \omega h$ sufficiently small we may write\(^1\)

$$r(ix) = e^{(-a+ib)x}$$

(17)

where $a = a(x)$, $b = b(x)$ are real-valued functions exhibiting the numerical damping ($a \neq 0$) and phase error ($b \neq 1$) of the single-step method generated by $r(z)$ in the approximation of (16). Extending the tables in Brusa and Nigro,\(^1\) we list in Table III the lowest order terms of the asymptotic expansions in $x$ of the local truncation error $|e^{ix} - r(ix)|$, of the ‘damping coefficient’

![Figure 1. Damping coefficient $|a(x/\nu)|$](image-url)
\[ |a| \] and of the normalized 'phase error' \[ |b - 1| \]. The entries of Table III may be obtained straightforwardly by expanding in Taylor series about \( x = 0 \) both sides of the equation (17) and equating coefficients; note the relation of the order of accuracy (even–odd) of a method to the lowest order terms in the expansions of \(|a|, |b - 1|\).

Table III. Lowest order terms of \(|e^{ix} - r(ix)|, |a|, |b - 1|\)

<table>
<thead>
<tr>
<th>Method</th>
<th>Eqn (13)</th>
<th>Eqn (14)</th>
<th>Eqn (15)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>(2, 3) RK</td>
<td>(3, 3) RK</td>
<td>(3, 4) RK</td>
</tr>
<tr>
<td>(</td>
<td>e^{ix} - r(ix)</td>
<td>)</td>
<td>0.089779x^4</td>
</tr>
<tr>
<td>(</td>
<td>a</td>
<td>)</td>
<td>0.089779x^3</td>
</tr>
<tr>
<td>(</td>
<td>b - 1</td>
<td>)</td>
<td>0.098113x^4</td>
</tr>
</tbody>
</table>

These expansions show the behaviour of the several types of error for every small \( x \). For moderate values of \( x \), and to afford some comparison among the methods, we offer the graphs of

Figure 2. Phase error \(|b(x/\nu) - 1|\)
Figures 1 and 2. We consider the functions

\[ |a(x)| = -\left(\ln |r(ix)|\right)/x \]
\[ |b(x) - 1| = \left| \frac{1}{x} \arctan \left( \frac{\text{Im}(r(ix))}{\text{Re}(r(ix))} \right) - 1 \right| \]

In order to facilitate the comparison among these single-step methods, and also to allow their comparison with standard linear multi-step methods, we assume that the work required of a \( \nu \)-stage single-step method for each time step is (essentially) \( \nu \) times that of a linear multi-step method. It follows that the comparison of damping and phase error should be made, for methods of commensurate work, according to \( |a(x/\nu)| \) and \( |b(x/\nu) - 1| \), that are precisely the quantities graphed versus \( x \) in Figures 1 and 2, respectively. In Figure 1, for moderate values of \( x \), the \((3, 3)\) method (14) shows the best behaviour.\(^6\) This is also true as far as its phase error is concerned (Figure 2). The \((3, 4)\) method is also competitive for moderate \( x \) due to the fact that \( b(x) = 1 \) for \( x = 0.686 \).

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REFERENCES