Wave scattering by an elastic obstacle with interior cuts

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Dedicated to Professor Frank-Olme Speck on the occasion of his 60th birthday

We consider the direct interaction problem describing the scattering of acoustic waves by an elastic obstacle with interior cracks. By the potential method we reduce the problem to an equivalent system of integral (pseudodifferential) equations and study its solvability. In particular, we formulate necessary and sufficient conditions for the solvability in terms of the so-called Jones modes. We show that the direct scattering problem is solvable for arbitrary values of the frequency parameter and for arbitrary incident wave functions if the crack surface is traction free. We also investigate the regularity of the displacement field, and analyze the stress singularities at the crack edge.

1 Introduction

Direct and inverse problems related to the interaction between vector fields of different dimension have received much attention in the mathematical and engineering scientific literature and have been intensively investigated in the past years. They arise in many physical and mechanical models describing the interaction of two different media.

Many authors have considered and studied in detail direct problems of the interaction between an elastic isotropic body which occupies a bounded region $\Omega^+$ (where a three-dimensional elastic vector field is to be defined), and some isotropic medium (fluid say) which occupies the unbounded exterior region, the complement of $\Omega^+$ (where a scalar scattered wave field is to be defined). The time-harmonic dependent unknown vector and scalar fields are coupled by some kinematic and dynamic conditions on the boundary $\partial\Omega^+$. Main attention has been given to the problems determining the manner in which an incoming acoustic wave is scattered by an isotropic elastic body immersed in a compressible inviscid fluid. An exhaustive information in this direction concerning theoretical and numerical results can be found in [1]–[4], [15], [17], [18], [22] and [30]. The case of anisotropic obstacle has been treated in [19], [26], [27] and [29]. In [26] the corresponding inverse problem is also considered. This kind of problems arise in detecting and identifying submerged objects, for instance. In [28], the case of Lipschitz domains was studied when the transmission conditions are understood in the sense of nontangential convergence almost everywhere.

The main goal of this paper is to investigate the direct problems related with the wave scattering by an elastic obstacle which has interior cracks and to give the qualitative description of the corresponding acoustic and elastic fields.

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As about the solution, we propose an ansatz of combinations of single and double layer potentials. By these special representation formulas based on the acoustic and elastic fields we reduce equivalently the original transmission problem to a system of integral equations.

Furthermore, by the use of the potential method we derive necessary and sufficient conditions for the solvability of the original transmission problem. In particular, we show that the direct scattering problems are solvable for arbitrary values of the frequency parameter and for arbitrary incident wave functions if the crack surface is traction free. It is established that the scalar radiating acoustic (pressure) field in the exterior domain is defined uniquely, while the elastic (displacement) vector field in the interior domain is defined modulo Jones modes, in general. Moreover, we obtain the $C^\alpha$-regularity property with $\alpha < 1/2$ for the displacement vector in a neighbourhood of the interior crack edges.

2 Formulation of the problem and uniqueness results

2.1 Basic field equations

Let $\Omega^+$ be a bounded domain in $\mathbb{R}^3$ and $\Omega^- = \mathbb{R}^3 \setminus \overline{\Omega^+}$ with $\Omega^+ = \Omega^+ \cup S$ where $S := \partial\Omega^+ = \partial\Omega^-$ is a smooth connected surface. The region $\Omega^+$ is occupied by an elastic material with the Lamé constants $\lambda$ and $\mu$, and density $\rho_1$. We denote by $u = (u_1, u_2, u_3)^T$ the corresponding elastic displacement field. The elastic body is immersed in a compressible inviscid fluid with density $\rho$, occupying the exterior domain $\Omega^-$. In the exterior domain we have a total acoustic wave function (pressure) which is the sum of an incident wave and a scattered wave, $u^{(total)} = u^{(inc)} + u^{(sc)}$.

We assume that the elastic obstacle has an interior crack $\Sigma$ which is a two-dimensional, two-sided subsurface of some closed surface $S_0$ surrounding a proper subdomain $\Omega_0$ of $\Omega^-$: $\Sigma \subset S_0$, $\Sigma = \Sigma \cup \partial\Sigma$, $S_0 = \partial\Omega_0$, $\Omega_0 = \Omega^+ \cap S_0 \cap S = \emptyset$. Further, let $\Omega_+ := \Omega^+ \setminus \Sigma$.

Throughout the paper, for simplicity, we assume that $S, S_0, \Sigma,$ and $\partial\Sigma$ are $C^\infty$-regular (if not otherwise stated). By $n(x)$ we denote the outward unit normal vector at the point $x \in S$ or $x \in S_0$; this defines the positive direction of the normal vector on the subsurface $\Sigma$ uniquely.

As we have mentioned in the Introduction, we will consider the time harmonic dependent dynamical process in both domains which is described by the so-called steady state oscillation equations.

The steady state oscillation equations of the classical elasticity read (see, e.g., [20])

$$A(\partial, \omega)u(x) \equiv \mu \Delta u(x) + (\lambda + \mu) \text{grad} \, \text{div} \, u(x) + g_1 \omega^2 u(x) = 0 \quad \text{in} \quad \Omega_+^\omega, \quad (2.1)$$

where $\omega > 0$ is the frequency parameter and the Lamé constants satisfy the inequalities: $\mu > 0$ and $3\lambda + 2\mu > 0$.

We denote the classical stress operator by $T(\partial, n) = [T_{kj}(\partial, n)]_{3 \times 3}$ and the corresponding stress vector acting on a surface element with the unit normal $n = (n_1, n_2, n_3)$ by $T(\partial, n)u$:

$$T(\partial, n)u = 2\mu \partial_n u + \lambda n \, \text{div} \, u + \mu [n \times \text{curl} \, u],$$

where $\partial_n = \partial / \partial n$ stands for the usual normal derivative and the symbol $\times$ denotes the cross product in $\mathbb{R}^3$.

In the exterior domain the scattered wave function solves the Helmholtz equation

$$a(\partial, \omega)w^{(sc)}(x) = \Delta w^{(sc)}(x) + g_2 \omega^2 w^{(sc)}(x) = 0 \quad \text{in} \quad \Omega^-,$$ \quad (2.2)

and at infinity satisfies the Sommerfeld radiation condition ([6] and [34])

$$\frac{\partial}{\partial |x|} w^{(sc)}(x) - i \kappa w^{(sc)}(x) = O(|x|^{-2}) \quad \text{as} \quad |x| \to \infty, \quad \kappa = \sqrt{g_2} \omega > 0.$$ \quad (2.3)

Such solutions will be referred to as radiating and we write $w^{(sc)} \in \text{Som}(\Omega^-)$.

Note that the incident wave function $w^{(inc)}$ is assumed to satisfy the Helmholtz equation (2.2) as well, except possibly at some places in $\Omega^-$, but $w^{(inc)}$ is not radiating, in general. In particular, plane waves or the fundamental function of the Helmholtz operator can be taken as incident wave functions.
2.2 Formulation of the transmission problem

We look for weak solutions $u$ and $w^{(sc)}$ to these differential equations (2.1)–(2.2) in the Sobolev (Bessel potential) spaces $[H_2^1(\Omega_+^\Sigma)]^3$ and $H_1^2(\partial(\Omega^-) \cap \text{Som}(\Omega^-))$, respectively. Note that, actually these solutions are $C^\infty$-regular in $\Omega_+^\Sigma$ and $\Omega_-$ due to the ellipticity of the corresponding differential operators. In a neighbourhood of the crack edge $\partial \Sigma$ solutions to Equation (2.1) do not possess $C^\alpha$-regularity with $\alpha > 1/2$, in general. More precise regularity properties will be established later. By standard arguments, for such weak solutions we can define the stress vector $T(\partial, n)u$ and the normal derivative $\partial_n w^{(sc)}$ as functionals belonging to the spaces $[H_2^{-\frac{4}{3}}(\Sigma)]^3 \times [H_2^{-\frac{4}{3}}(\Sigma)]^3$ and $H_2^{-\frac{4}{3}}(\Sigma)$, respectively.

The elastic field and the total wave function are coupled by the so-called kinematic and dynamic conditions on the boundary $\partial \Omega^+$ (see, e.g., [1] and [17]–[18]). They have to satisfy the following transmission conditions on the interface $S$:

\[
\begin{align*}
[u(x) \cdot n(x)]^+ &= b_1 [\partial_n w^{(tot)}(x)]^- + f_0(x), \\
[T(\partial, n)u(x)]^+ &= b_2 n(x) [w^{(tot)}(x)]^- + F(x),
\end{align*}
\]

where

\[
\begin{align*}
f_0 &= b_1 [\partial_n w^{(inc)}(x)]^- \in H_2^{-\frac{4}{3}}(S), \quad b_1 = -1, \quad b_2 = \left[\rho_2 \omega^2\right]^{-1}, \\
F &= (F_1, F_2, F_3)^\top = b_2 n(x) [w^{(inc)}(x)]^- \in \left[H_2^{-\frac{4}{3}}(S)\right]^3.
\end{align*}
\]

Here and in what follows the symbols $[\cdot]^\pm = [\cdot]_{\partial \Omega^\pm}$ denote the limits on $\partial \Omega^\pm$ from $\Omega^\pm$.

Note that all the results below are valid if $b_1$ and $b_2$ are complex numbers such that the product $\overline{b_1}b_2$ is real and different from zero. Here, and in what follows, the “overbar” symbol denotes the complex conjugate and “dot” denotes the “real” scalar product of complex vectors, $c' \cdot c'' = \sum_{j=1}^3 c'_j c''_j$ for $c' = (c'_1, c'_2, c'_3)$ and $c'' = (c''_1, c''_2, c''_3)$ with $c'_j, c''_j \in \mathbb{C}$.

Further, we assume that $u$ satisfies the following inhomogeneous boundary conditions on the crack surface $\Sigma$:

\[
[T(\partial, n)u]^+ = F^+, \quad [T(\partial, n)u]^- = F^- \quad \text{on} \quad \Sigma.
\]

Actually, instead of these conditions we consider the following equivalent version,

\[
\begin{align*}
[T(\partial, n)u]^+ - [T(\partial, n)u]^- &= F^+ - F^- \quad \text{on} \quad \Sigma, \\
[T(\partial, n)u]^+ + [T(\partial, n)u]^- &= F^+ + F^- \quad \text{on} \quad \Sigma.
\end{align*}
\]

Here

\[
r_\Sigma F^\pm = r_\Sigma(F_1^\pm, F_2^\pm, F_3^\pm)^\top \in \left[H_2^{-\frac{4}{3}}(\Sigma)\right]^3, \quad F^+ + F^- \in \left[\tilde{H}_2^{-\frac{4}{3}}(\Sigma)\right]^3,
\]

where $H_2^2(\Sigma)$ is the space of restrictions to $\Sigma$ of functions from the space $H_2^2(S_0)$, while

\[
\tilde{H}_2^2(\Sigma) := \{ f \in H_2^2(S_0) : \text{supp} f \subset \Sigma \} \quad \text{for} \quad s \in \mathbb{R},
\]

and $r_\Sigma$ denotes the restriction operator to $\Sigma$. We recall that $\tilde{H}_2^2(\Sigma)$ and $H_2^{-\frac{4}{3}}(\Sigma)$ are mutually adjoin function spaces (for details see, e.g., [21] and [32]).

Thus, our problem is to find a weak solution pair, an elastic vector field of displacements $u \in [H_2^1(\Omega_+^\Sigma)]^3$ and a radiating scalar scattered field $w^{(sc)} \in H_2^2(\partial(\Omega^-) \cap \text{Som}(\Omega^-))$, satisfying the differential equations (2.1) and (2.2) in the distributional sense, and the transmission conditions (2.4)–(2.5) and the boundary conditions (2.8) in the functional sense (cf. e.g. [21]). We will refer to this transmission problem as the Problem (TP).
2.3 Jones modes and Jones eigenfrequencies. Uniqueness theorem

We denote by $J(Ω_+)$ the set of values of the frequency parameter $ω > 0$ for which the following homogeneous boundary value problem

$$A(∂, ω) u = 0 \quad \text{in} \quad Ω_+^c,$$

$$[T(∂, n) u]^+ = 0, \quad [u \cdot n]^+ = 0 \quad \text{on} \quad Σ,$$

$$[T(∂, n) u]^+ = 0, \quad [T(∂, n) u]^− = 0 \quad \text{on} \quad Σ,$$

admits a nontrivial solution $u ∈ [H_1^2(Ω_+)]^3$. Such solution vectors are called Jones modes, while the corresponding values of $ω$ are called Jones eigenfrequencies. We denote by $X_ω(Ω_+)$ the space of Jones modes corresponding to $ω$. Note that $J(Ω_+^c)$ is at most enumerable, and for each $ω ∈ J(Ω_+^c)$ the space of associated Jones modes is of finite dimension. Clearly, if $u ∈ X_ω(Ω_+^c)$, then also $−u ∈ X_ω(Ω_+^c)$.

The result about the uniqueness of solution for the homogeneous direct problem (TP) (having $f_0 = 0$, $F = 0$, and $E^± = 0$ in (2.4), (2.5), and (2.7)) is given by the following assertion (where we use the notation $∃ c$ for the imaginary part of a complex number $c$).

**Theorem 2.1** Let a pair $(u, w^{(sc)}) ∈ [H_1^2(Ω_+)]^3 × [H_1^2, Ω(Ω^−)]$ be a solution of the homogeneous transmission problem (TP), where $∥ b_1 ∥ ≠ 0$ and $∃ [∥ b_1 ∥] = 0$. Then $w^{(sc)} = 0$ in $Ω^−$ and $u ∈ X_ω(Ω_+^c)$.

**Proof.** Let $(u, w^{(sc)})$ have the properties stated in the theorem. Further, let $R > 0$ be a sufficiently large number and $Ω_{R} = Ω^− ∩ B(R)$ where $B(R)$ is a ball centered at the origin and radius $R$. By Green’s formulas we then have (16), (11) and (20)

$$\int_{Ω_{R}} \left[ ∇ w^{(sc)} \right]^2 - \varrho_1 ω^2 |w^{(sc)}|^2 \, dx =$$

$$= - \left( \partial_n w^{(sc)} \right)^T \left[ \frac{w^{(sc)}}{S} \right]_S + \int_{∂ B(R)} \partial_n w^{(sc)} w^{(sc)} \, dS,$$

$$\int_{Ω_+^c} \left[ E(u, w^{(sc)}) - \varrho_1 ω^2 |u|^2 \right] \, dx = \left( [Tu]^+_S, [w^{(sc)}]_S \right)_S + \left( [Tu]^+_S - [Tu]^−_S \right) \left( \frac{w^{(sc)}}{S} \right)_S +$$

$$+ \left( [Tu]^−_S, \left( \frac{w^{(sc)}}{S} \right)_S \right)_S,$$

where $∇ = (∂_1, ∂_2, ∂_3)$, $E(u, w^{(sc)})$ is a nonnegative quadratic form with respect to the elements of the strain tensor $ε_{kj} = 2^{-1}(∂_j u_k + ∂_k u_j)$,

$$E(u, w^{(sc)}) = \sum_{k,j=1}^{3} σ_{kj} \frac{w^{(sc)}}{S} \geq c_0 \sum_{k,j=1}^{3} |ε_{kj}|^2, \quad σ_{kj} = δ_{kj} (ε_{11} + ε_{22} + ε_{33}) + 2με_{kj},$$

with some positive constant $c_0$. Here $δ_{kj}$ is the Kronecker delta. The symbols $(·, ·)_S$ and $(·, ·)_Σ$ are the duality brackets between the dual spaces $[H_2^2(S)]^m$ and $[H_2^2(Σ)]^m$, or $[H_2^2(S)]^m$ and $[H_2^2(Σ)]^m$, or $[H_2^2(Σ)]^m$ and $[H_2^2(Σ)]^m$. Here $m ≥ 1$ is a positive integer. For regular functions, e.g., $f, g ∈ [L_2(M)]^m$, we have

$$\langle f, g \rangle_M = \int_M \sum_{k=1}^{m} f_k g_k \, dM, \quad M ∈ \{S, Σ\}.$$

Due to the inclusions

$$[Tu]^+_S ∈ [H_2^1(Σ)]^3, \quad \left[ u \right]^+_S ∈ [H_2^1(Σ)]^3,$$

$$[Tu]^+_S - [Tu]^−_S ∈ r_Σ \left[ H_2^1(S) \right]^3, \quad \left[ u \right]^+_S - \left[ u \right]^−_S ∈ r_Σ \left[ H_2^1(S) \right]^3,$$
the above dualities are correctly defined (considering the possibility of extending by zero the last two elements).

Since \([Tu]_S^+ \cdot [\Pi]_S^+ = \frac{1}{b_1} b_2 w^{(sc)} \big|_S \left[ \partial_n w^{(sc)} \right] \big|_S^+\), \([Tu]_C^+ = 0\), and \(b_1 b_2\) is a real number different from zero, from the above formulas we get

\[ \Im \int_{\partial B(R)} w^{(sc)} \partial_n w^{(sc)} dS = -\textrm{Re} \int_{\partial B(R)} |w^{(sc)}|^2 dS + \mathcal{O}(R^{-1}) = 0, \]

due to the radiating property of the scattered field. Whence \(\lim_{R \to \infty} \int_{\partial B(R)} |w^{(sc)}|^2 dS = 0\), and \(w^{(sc)} = 0\) in \(\Omega^-\) by the celebrated Rellich–Vekua lemma ([6] and [34]).

From the homogeneous conditions (2.4), (2.5), and (2.7) it then follows that \(u \in X_\omega(\Omega^+_\omega)\), which completes the proof. \(\Box\)

**Corollary 2.2** If \(\omega \notin J(\Omega^+_\omega)\), then the homogeneous transmission problem (TP) possesses only the trivial solution.

### 3 Existence and regularity results

#### 3.1 Potentials and boundary integral operators

Here we recall some well-known properties of the potential type operators. Let us denote the single and double layer potentials corresponding to the differential operators (2.1) and (2.2) by

\[ V_{\omega, \varphi}(y) = \int_S \Gamma(x - y, \omega) \varphi(x) dS, \]
\[ W_{\omega, \varphi}(y) = \int_S [\Gamma(\partial_y, n(y)) \Gamma(x - y, \omega)]^T \varphi(x) dS, \]

\[ \Gamma_{\omega, \varphi}(y) = \int_S \gamma(x - y, \omega) \varphi(x) dS, \]
\[ W_{\omega, \varphi}(y) = \int_S [\partial_n(y) \gamma(x - y, \omega)] \varphi(x) dS, \]

where \(x \in \mathbb{R}^3 \setminus S\), \(g = (g_1, g_2, g_3)^T\) and \(g_4\) are densities of the potentials, \(\Gamma(\cdot, \omega)\) is the Kupradze’s fundamental matrix of the operator \(A(\partial, \omega)\) (cf. [20, Ch. 3]) and \(\gamma(\cdot, \omega)\) is the fundamental function of the Helmholtz operator, i.e., \(A(\partial, \omega) \Gamma(x, \omega) = I_3 \delta(x)\) and \(a(\partial, \omega) \gamma(x, \omega) = \delta(x)\),

\[ \Gamma(x, \omega) := \begin{bmatrix} \Gamma_{kj}(x, \omega) \end{bmatrix}_{j=1}^{3 \times 3}, \quad \Gamma_{kj}(x, \omega) = \sum_{i=1}^{2} \left( \delta_{kl} \alpha_l + \beta_l \frac{\partial^2}{\partial x_k \partial x_j} \right) \frac{\exp \{ik_j |x|\}}{|x|}, \]
\[ \gamma(x, \omega) := -\frac{1}{4\pi} \exp \{i\varphi |x|\} |x|; \]

here \(\delta(\cdot)\) is the Dirac distribution, \(\delta_{pq}\) is the Kronecker delta, \(\varphi\) is given by (2.3), and

\[ k_1^2 = g_1 \omega^2 + \frac{2}{\lambda + 2\mu}, \quad k_2^2 = g_2 \omega^2, \quad \alpha_l = -\frac{\delta_{2l}}{4\pi \mu}, \quad \beta_l = \frac{(-1)^{l+1}}{4\pi g_1 \omega^2}, \quad l = 1, 2. \]  

(3.2)

Further we introduce the boundary operators generated by the above potentials

\[ (\mathcal{H}_{\omega, \varphi}) (x) = \int_S \Gamma(x - y, \omega) g(y) dS, \quad x \in S, \]  

(3.3)

\[ (\mathcal{K}_{\omega, \varphi}^{(1)}) (x) = \int_S T(\partial_x, n(x)) \Gamma(x - y, \omega) g(y) dS, \quad x \in S, \]  

(3.4)
We look for the solution pair of our transmission problem (TP) in the form

\[(\mathcal{K}_{A,S}^{(2)} g)(x) = \int_S [T(\partial_y, n(y))\Gamma(y - x, \omega)]^T g(y) \, dS, \quad x \in S, \quad (3.5)\]

\[(\mathcal{L}_{A,S}^+ g)(x) = \lim_{\Omega_2 \ni x \to x \in S} T(\partial_x, n(x))W_{A,S}(g)(z), \quad x \in S.\]

It is well-known that there hold the jump relations (see [Appendix, Theorems A.1 and A.2])

\[[V_{A,S}(g)]_S^\pm = \mathcal{H}_{A,S} g,\]

\[[T(\partial_n, n)V_{A,S}(g)]_S^\pm = \left[ \mp 2^{-1}I_3 + \mathcal{K}_{A,S}^{(1)} \right] g,\]

\[[W_{A,S}(g)]_S^\pm = \left[ \pm 2^{-1}I_3 + \mathcal{K}_{A,S}^{(2)} \right] g,\]

\[\mathcal{L}_{A,S}^+ g := \mathcal{L}_{A,S}^- g = \mathcal{L}_{A,S}^-,\]

where \(I_3\) stands for the unit 3 × 3 matrix.

Note that \(\mathcal{H}_{A,S}\) is an integral operator with weakly singular kernel, \(\mathcal{K}_{A,S}^{(1)}\) and \(\mathcal{K}_{A,S}^{(2)}\) are singular integral operators, and \(\mathcal{L}_{A,S}\) is a singular integro-differential operator.

The potentials \(V_{A,S}\) and \(W_{A,S}\), and the corresponding operators \(\mathcal{H}_{A,S}, \mathcal{K}_{A,S}^{(1)}, \mathcal{K}_{A,S}^{(2)}, \) and \(\mathcal{L}_{A,S}\) are defined analogously (with \(\Sigma\) instead of \(S\)).

Quite similarly, the potentials \(V_{a,S}\) and \(W_{a,S}\) given by (3.1), generate the scalar boundary operators \(\mathcal{H}_{a,S}, \mathcal{K}_{a,S}^{(1)}, \mathcal{K}_{a,S}^{(2)},\) and \(\mathcal{L}_{a,S}\),

\[(\mathcal{H}_{a,S} g_4)(x) = \int_S \gamma(x - y, \omega)g_4(y) \, dS, \quad x \in S,\]

\[(\mathcal{K}_{a,S}^{(1)} g_4)(x) = \int_S [\partial_n(x)\gamma(x - y, \omega)]g_4(y) \, dS, \quad x \in S,\]

\[(\mathcal{K}_{a,S}^{(2)} g_4)(x) = \int_S [\partial_n(y)\gamma(y - x, \omega)]g_4(y) \, dS, \quad x \in S,\]

\[(\mathcal{L}_{a,S}^+ g_4)(x) = \lim_{\Omega_2 \ni z \to x \in S} \nabla_n W_{a,S}(g_4)(z) \cdot n(x), \quad x \in S.\]

For these scalar potentials we have the similar jump relations (see Appendix)

\[[V_{a,S}(g_4)]_S^\pm = \mathcal{H}_{a,S} g_4,\]

\[[\partial_n V_{a,S}(g_4)]_S^\pm = \left[ \mp 2^{-1}I + \mathcal{K}_{a,S}^{(1)} \right] g_4,\]

\[[W_{a,S}(g_4)]_S^\pm = \left[ \pm 2^{-1}I + \mathcal{K}_{a,S}^{(2)} \right] g_4,\]

\[\mathcal{L}_{a,S}^+ g_4 := \mathcal{L}_{a,S}^- g_4 = \mathcal{L}_{a,S}^-,\]

where \(I\) stands for the unit operator.

Mapping and Fredholm properties of the above boundary operators in various Hölder, Sobolev–Slobodetskiǐ, Bessel potential, and Besov function spaces are described in Appendix.

### 3.2 Reduction to boundary integral equations and existence results

We look for the solution pair of our transmission problem (TP) in the form

\[u(x) = V_{A,S}(g)(x) + W_{A,S}(\psi)(x) + V_{A,S}(\varphi)(x), \quad x \in \Omega^+_2, \quad (3.6)\]

\[w^{(sc)}(x) = W_{a,S}(g_4)(x) - i V_{a,S}(g_4)(x), \quad x \in \Omega^-, \quad (3.7)\]

where \(g = (g_1, g_2, g_3)^\top, g_4, \psi = (\psi_1, \psi_2, \psi_3)^\top, \) and \(\varphi = (\varphi_1, \varphi_2, \varphi_3)^\top\) are unknown densities in the following framework:

\[g \in \left[H_2^\frac{3}{2}(S)\right]^3, \quad g_4 \in \left[H_2^\frac{3}{2}(S)\right]^3, \quad \psi \in \left[H_2^\frac{3}{2}(\Sigma)\right]^3, \quad \varphi \in \left[H_2^\frac{3}{2}(\Sigma)\right]^3.\]
These inclusions imply that \( u \in \mathcal{H}^1_0(\Omega^+) \) and \( w^{(\text{sc})} \in H^1_{2,\text{loc}}(\Omega^-) \cap \text{Som}(\Omega^-) \) (see [Appendix, Theorem A.2]).

Taking into account the transmission conditions (2.4) and (2.5) on \( S \) and the boundary conditions (2.8) on \( \Sigma \), and also with the help of the above mentioned jump relations, we arrive at the linear system of integral equations

\[
\left( -2^{-1}I_3 + \mathcal{K}^{(1)}_{a,S} \right) g - b_2 n D_{a,S} g_4 + \left[ TW_{A,S}(\psi) \right]_S + \left[ TV_{A,S}(\varphi) \right]_S = F \quad \text{on } S, \quad (3.8)
\]

\[
(\mathcal{H}_{a,S} g) \cdot n - b_1 N_{a,S} g_4 + \left[ W_{A,S}(\psi) \right]_S \cdot n + \left[ V_{A,S}(\varphi) \right]_S \cdot n = f_0 \quad \text{on } S, \quad (3.9)
\]

\[
\left[ TV_{A,S}(g) \right]_S + \mathcal{L}_{A,S} \psi + \mathcal{K}^{(1)}_{A,S} \varphi = 2^{-1}(F^+ + F^-) \quad \text{on } \Sigma, \quad (3.10)
\]

\[
\varphi = F^- - F^+ \quad \text{on } \Sigma, \quad (3.11)
\]

where the operators

\[
D_{a,S} g_4 := \left[ -2^{-1}I + \mathcal{K}^{(2)}_{a,S} \right] g_4 - i \mathcal{H}_{a,S} g_4, \quad N_{a,S} g_4 := \mathcal{L}_{a,S} g_4 - i \left[ 2^{-1}I + \mathcal{K}^{(1)}_{a,S} \right] g_4
\]

are generated by the exterior limiting values on \( S \) of the linear combination of the layer potentials (3.7) and its normal derivative.

Here, the symbols \([ \cdot ]_S\) and \([ \cdot ]_\Sigma\) denote the restrictions of the corresponding functions on \( S \) and \( \Sigma \) respectively (and we recall that for these restriction operators we also employ the notation \( r_S \) and \( r_\Sigma \)).

We introduce the function spaces

\[
X := \left[ H^{\frac{1}{2}}(S) \right]^3 \times \left[ H^{\frac{3}{2}}(\Sigma) \right]^3 \times \left[ \bar{H}^{\frac{1}{2}}(\Sigma) \right]^3,
\]

\[
Y := \left[ H^{\frac{1}{2}}(S) \right]^3 \times \left[ H^{\frac{1}{2}}(S) \right]^3 \times \left[ H^{\frac{1}{2}}(\Sigma) \right]^3 \times r_\Sigma \left[ \bar{H}^{\frac{1}{2}}(\Sigma) \right]^3.
\]

Further, let us introduce the ten-dimensional vector function of unknown densities \( G := (g, g_4, \psi, \varphi)^T \). We require that \( G \in X \). Moreover, let us construct the ten-dimensional vector function by the right-hand side expressions of the system (3.8)–(3.11), \( Q := (F, f_0, (F^+ + F^-)/2, F^- - F^+)^T \). It is evident that \( Q \in Y \), due to the inclusions (2.6) and (2.9).

Denote the 10 × 10 matrix operator generated by the left-hand side expressions of the simultaneous integral equations (3.8)–(3.11) by \( \mathcal{P} \) and rewrite these equations as

\[
\mathcal{P} G = Q.
\]

The operator \( \mathcal{P} \) can be written as the following 10 × 10 matrix operator:

\[
\mathcal{P} := \begin{bmatrix}
    r_S \left[ -2^{-1}I_3 + \mathcal{K}^{(1)}_{A,S} \right] & r_S \left[ TW_{A,S} \right] & r_S \left[ TV_{A,S} \right] \\
    r_S \left[ n_k {\mathcal{H}_{A,S}}_{k,j,1} \right] & r_S \left[ n_k {\mathcal{W}_{A,S}}_{k,j} \right] & r_S \left[ n_k {\mathcal{V}_{A,S}}_{k,j} \right] \\
    r_S \left[ 0 \right] & r_S \left[ 0 \right] & r_S \left[ 0 \right] \\
\end{bmatrix}
\]

In what follows we investigate the mapping and Fredholm properties of the operator \( \mathcal{P} \).

**Theorem 3.1** The operator

\[
\mathcal{P} : X \longrightarrow Y
\]

is Fredholm with zero index.

**Proof.** The mapping property (3.13) follows immediately from the mapping properties of the boundary integral (pseudodifferential) operators involved in the expressions of the entries of \( \mathcal{P} \) (see [Appendix, Theorem A.2]).
To study the Fredholm property and the Fredholm index of the operator $P$ in (3.13) we proceed as follows. Let us consider the diagonal $10 \times 10$ matrix operator:

$$
T := \begin{pmatrix}
    r_5 
    \begin{bmatrix}
        -2^{-1}I_3 + \mathcal{K}_{A,s}^{(1)} & -i \mathcal{H}_{A,s} & -i \mathcal{H}_{A,s} \\
        -i \mathcal{H}_{A,s} & r_s[0]_{3\times3} & r_s[0]_{3\times3} & r_s[0]_{3\times3} \\
        -i \mathcal{H}_{A,s} & r_s[0]_{3\times3} & r_s[0]_{3\times3} & r_s[0]_{3\times3} \\
        -i \mathcal{H}_{A,s} & r_s[0]_{3\times3} & r_s[0]_{3\times3} & r_s[0]_{3\times3} \\
        r_s[0]_{3\times3} & r_s[0]_{3\times3} & r_s[0]_{3\times3} & r_s[0]_{3\times3}
    \end{bmatrix}
\end{pmatrix}.
$$

The operator

$$
-2^{-1}I_3 + \mathcal{K}_{A,s}^{(1)} - i \mathcal{H}_{A,s} : \begin{pmatrix} H_{2,\mathfrak{H}}^{\frac{1}{2}}(S) \end{pmatrix}^3 \rightarrow \begin{pmatrix} H_{2,\mathfrak{H}}^{\frac{1}{2}}(S) \end{pmatrix}^3
$$

corresponds to the interior Robin type boundary value problem for the domain $\Omega^+$ (with the boundary condition $[Tu]^+ - [u]^+$ prescribed on $S$) and is invertible (cf. [Appendix, Theorem A.2]). The operators

$$
\mathcal{N}_{A,s} : H_{2,\mathfrak{H}}^{\frac{1}{2}}(S) \rightarrow H_{2,\mathfrak{H}}^{\frac{1}{2}}(S), \quad \mathcal{L}_{A,\Sigma} : \begin{pmatrix} H_{2,\mathfrak{H}}^{\frac{1}{2}}(\Sigma) \end{pmatrix}^3 \rightarrow \begin{pmatrix} H_{2,\mathfrak{H}}^{\frac{1}{2}}(\Sigma) \end{pmatrix}^3
$$

are invertible as well (see [Appendix, Theorems A.3 and A.5]). Therefore, the diagonal operator $T : X \rightarrow Y$ is invertible.

It is easy to verify that the operator $P - T : X \rightarrow Y$ is compact. Therefore, the operator (3.13) is Fredholm with index equal to zero.

To establish necessary and sufficient conditions for the solvability of Equation (3.12), we start by considering the corresponding adjoint operator $P^\dagger : Y^* \rightarrow X^*$, where

$$
Y^* = \begin{pmatrix} H_{2,\mathfrak{H}}^{\frac{1}{2}}(S) \end{pmatrix}^3 \times H_{2,\mathfrak{H}}^{\frac{1}{2}}(S) \times \begin{pmatrix} \tilde{H}_{2,\mathfrak{H}}^{\frac{1}{2}}(\Sigma) \end{pmatrix}^3 \times \begin{pmatrix} H_{2,\mathfrak{H}}^{\frac{1}{2}}(\Sigma) \end{pmatrix}^3,
$$

$$
X^* = \begin{pmatrix} H_{2,\mathfrak{H}}^{\frac{1}{2}}(S) \end{pmatrix}^3 \times H_{2,\mathfrak{H}}^{\frac{1}{2}}(S) \times \begin{pmatrix} H_{2,\mathfrak{H}}^{\frac{1}{2}}(\Sigma) \end{pmatrix}^3 \times \begin{pmatrix} H_{2,\mathfrak{H}}^{\frac{1}{2}}(\Sigma) \end{pmatrix}^3.
$$

Note that $\langle P^\dagger G, G^* \rangle = \langle G, P^\dagger G^* \rangle$, for all $G \in X$, and for all $G^* \in Y^*$, where in the left-hand side we have duality brackets between the spaces $Y$ and $Y^*$, while in the right-hand side we have duality brackets between the spaces $X$ and $X^*$.

Let $G^* = (g^*, g_4^*, \psi^*, \varphi^*)^\dagger \in Y^*$ and consider the homogeneous equation

$$
P^\dagger G^* = 0. \quad (3.14)
$$

The adjoint system (3.14) can be written componentwise as

$$
\begin{aligned}
    -2^{-1}I_3 + \mathcal{K}_{A,s}^{(2)} &+ \mathcal{H}_{A,s} + \mathcal{W}_{A,s}(\psi^*) + [W_{A,s}(\psi^*],_s = 0 \quad &\text{on} \quad S, \\
    -b_2 \mathcal{D}_{A,s}^\dagger (g^* \cdot n) - b_1 \mathcal{N}_{A,s}^\dagger g_4^* &+ 0 \quad &\text{on} \quad S, \\
    [TW_{A,s}(g^*)]_\Sigma + \mathcal{L}_{A,\Sigma} \psi^* &+ 0 \quad &\text{on} \quad \Sigma, \\
    [W_{A,s}(g^*)]_\Sigma &+ \mathcal{K}_{A,s}^{(2)} \psi^* &+ \varphi^* = 0 \quad &\text{on} \quad \Sigma.
\end{aligned} \quad (3.15) - (3.18)
$$

Here

$$
\mathcal{D}_{A,s}^\dagger := \begin{bmatrix} -2^{-1}I + \mathcal{K}_{A,s}^{(1)} \end{bmatrix} - i \mathcal{H}_{A,s} , \quad \mathcal{N}_{A,s}^\dagger := \mathcal{L}_{A,\Sigma} = i [2^{-1}I + \mathcal{K}_{A,s}^{(2)}].
$$

To find the boundary value problem corresponding to the system (3.15)–(3.18) let us introduce the following linear combinations of the layer potentials

$$
u^*(x) = W_{A,s}(g^*)(x) + V_{A,s}(g_4^* n)(x) + W_{A,s}(\psi^*)(x), \quad x \in \Omega^- \cup \Omega^+,
$$

$$w^*(x) = -b_2 V_{A,s}(g^* \cdot n)(x) - b_1 W_{A,s}(g_4^*)(x), \quad x \in \Omega^+ \cup \Omega^-.
$$
Evidently $u^*$ and $w^*$ are solutions of the differential equations (2.1) and (2.2), respectively. Moreover, $u^*$ satisfies the Sommerfeld–Kupradze radiation conditions at infinity (see [Appendix, §A.1]).

It is evident that the integral equations (3.14)–(3.18) are equivalent to the following boundary conditions for $u^*$ and $w^*$

\[
[u^*]_S^- = 0 \quad \text{on} \quad S,
\]

\[
[(\partial_n - i)w^*]_S^- = 0 \quad \text{on} \quad S,
\]

\[
[Tu^*]^+ [- Tu^*]^- = 0 \quad \text{on} \quad \Sigma,
\]

\[
\varphi^* = -2^{-1}\{[w^*]^+ + [w^*]^−\} \equiv - \{[W_{A,\Sigma}(U^*)]_\Sigma^- + [V_{A,\Sigma}(g^*n)]_\Sigma^- + K^{(2)}_{A,\Sigma}\psi^*\} \quad \text{on} \quad \Sigma.
\]

Since $u^*$ is a radiating vector function, the homogeneous Dirichlet type condition (3.19) implies that $u^*(x) = 0$ in $\Omega^- \,(\text{see Appendix}).$

Additionally, the homogeneous Robin type condition (3.20) yields $w^*(x) = 0$ in $\Omega^+$ due to the corresponding uniqueness theorem which is an easy consequence of Green’s formula

\[
\int_{\Omega^+} \{[\nabla w^*]^2 - \varrho_2\omega^2 [w^*]^2\} \, dx = \left(\frac{\partial w^*}{\partial n}\right)_S^+ - \left\langle \left[\frac{w^*}{\partial n}\right]_S^- , \left[w^*\right]^S \right\rangle.
\]

From these results and the jump relations

\[
[u^*]^+ - [u^*]^- = g^* \quad \text{on} \quad S,
\]

\[
[Tu^*]^+ - [Tu^*]^- = -g^* n \quad \text{on} \quad S,
\]

\[
[w^*]^+ - [w^*]^− = b_1 g^*_1 \quad \text{on} \quad S,
\]

\[
[\partial_\nu w^*]^+ - [\partial_\nu w^*]^− = -b_2 (g^* \cdot n) \quad \text{on} \quad S,
\]

we derive the following transmission and boundary conditions for the vector $u^*$ and the function $w^*$

\[
[u^* \cdot n]^+ = -b_2^{-1}[\partial_\nu w^*]^− \quad \text{and} \quad [Tu^*]^+ = -b_1^{-1}n[w^*]^− \quad \text{on} \quad S,
\]

\[
[Tu^*]^+ = 0, \quad [Tu^*]^- = 0 \quad \text{on} \quad \Sigma.
\]

These conditions yield that $u^*$ and $w^*$ solve the homogeneous transmission problem (TP) with the constants $-b_2^{-1}$ and $-b_1^{-1}$ in the place of $b_1$ and $b_2$, respectively (see the formulation of the original transmission problem in Subsection 2.2). By Theorem 2.1 we then conclude that $u^* = 0$ in $\Omega^-$ and either $u^* \in X_\Omega^\Omega^\Sigma$ if $\omega \in J(\Omega^\Sigma)$, or $u^* = 0$ in $\Omega^\Sigma$ if $\omega \notin J(\Omega^\Sigma)$.

Therefore, from (3.22) and (3.23), we have $g^*_1 = 0$ on $S$ and $g^* \cdot n = 0$ on $S$ pointwise in the sense of almost everywhere.

Further, we note that from (3.21) we get $g^* = [u^*]^+$ on $S$, which shows that $g^*$ coincides with the restriction of some Jones mode $u^*$ on the boundary $S$ if $\omega \in J(\Omega^\Sigma)$, while $g^* = 0$ on $S$ if $\omega \notin J(\Omega^\Sigma)$. It is also clear that the vectors $\psi^*$ and $\varphi^*$ can be expressed in terms of the vector $u^*$ if $\omega \in J(\Omega^\Sigma)$;

\[
\psi^* = [u^*]^+ - [u^*]^− \quad \text{and} \quad \varphi^* = -2^{-1}\{[u^*]^+ + [u^*]^−\} \quad \text{on} \quad \Sigma.
\]

Evidently, if $\omega \notin J(\Omega^\Sigma)$ then $\psi = \varphi = 0$ on $\Sigma$.

A more detailed analysis leads also to the following assertion.

**Lemma 3.2** The following equalities hold:

\[
\dim \ker P^\perp = \dim X_\omega(\Omega^\Sigma) = \dim \ker P.
\]

**Additionally:**

(i) **For an arbitrary Jones mode $u^*$ the vector $G^* = (g^*, g^*_2, \psi^*, \varphi^*)^\perp$**

\[
g^* = [u^*]^+, \quad g^*_2 = 0 \quad \text{on} \quad S,
\]

$$\psi^* = [u^*]^+_{\Sigma} - [u^*]^-_{\Sigma}, \quad \varphi^* = -2^{-1}\{[u^*]^+_{\Sigma} + [u^*]^-_{\Sigma}\} \quad \text{on} \quad \Sigma,$$

belongs to the null space of the operator $\mathcal{P}^\top$.

(ii) For an arbitrary element $G^* = (g^*, g^*_4, \psi^*, \varphi^*)^\top$ of the null space $\ker \mathcal{P}^\top$ the vector

$$u^*(x) = W_{A,s}(g^*)(x) + W_{A,\Sigma}(\psi^*)(x), \quad x \in \Omega^+_\Sigma,$$

(3.24)

represents a Jones mode.

(iii) The above correspondences preserve the linear independency.

Proof. First of all let us note that $\dim \ker \mathcal{P}^\top = \dim \ker \mathcal{P}$ due to Theorem 3.1. Further, let $\omega$ be a Jones mode of the rank $m$, i.e., $\dim \mathcal{X}_\omega(\Omega^+_\Sigma) = m$. Denote the linearly independent Jones modes by $v^{(1)}, \ldots, v^{(m)}$.

Due to the definition of Jones modes, we have

$$\left[ T(\partial, n)v^{(k)} \right]_{\Sigma}^+ = 0, \quad [v^{(k)} \cdot n]_{\Sigma}^+ = 0, \quad [T(\partial, n)v^{(k)}]^\bot_{\Sigma} = 0. \quad (3.25)$$

For an arbitrary solution $u \in \left[ H^1_0(\Omega^+_\Sigma) \right]^3$ of Equation (2.1) we have the following integral representation in $\Omega^+_\Sigma$

$$u(x) = \int_{\Sigma} \left[ T(\partial_y, n(y))\Gamma(y - x, \omega) \right]^\top \{[u]_{\Sigma}^+ \} dS - \int_{\Sigma} \Gamma(y - x, \omega) \{[Tu]_{\Sigma}^+ \} dS + \int_{\Sigma} \left[ T(\partial_y, n(y))\Gamma(y - x, \omega) \right]^\top \{[u]_{\Sigma}^+ - [u]_{\Sigma}^- \} dS$$

$$- \int_{\Sigma} \Gamma(y - x, \omega) \{[Tu]_{\Sigma}^+ - [Tu]_{\Sigma}^- \} dS, \quad x \in \Omega^+_\Sigma.$$

This formula can be obtained by standard arguments from Green’s identities (cf. [20]).

Therefore, in accordance with the homogeneous conditions (3.25) for the vectors $v^{(k)}$ we get the representation

$$v^{(k)}(x) = W_{A,s}(g^{(k)})(x) + W_{A,\Sigma}(\psi^{(k)})(x), \quad x \in \Omega^+_\Sigma,$$

(3.26)

where

$$g^{(k)} := [v^{(k)}]_{\Sigma}^+, \quad \psi^{(k)} := [v^{(k)}]_{\Sigma}^+ - [v^{(k)}]_{\Sigma}^-.$$

(3.27)

Let us show that the vector $G^{(k)} = \left(g^{(k)}; g_4^{(k)}, \psi^{(k)}, \varphi^{(k)}\right)^\top$ with $g^{(k)}$ and $\psi^{(k)}$ as in (3.27), and

$$g_4^{(k)} = 0 \quad \text{on} \quad S, \quad \varphi^{(k)} := -2^{-1}\{[v^{(k)}]_{\Sigma}^+ + [v^{(k)}]_{\Sigma}^\bot\} \quad \text{on} \quad \Sigma,$$

(3.28)

belongs to the null space of the operator $\mathcal{P}^\top$. Thus we have to check that (cf. (3.15)–(3.18))

$$-b_2^2D^\top_{a,s}(g^{(k)} \cdot n) - b_1N_{a,s}g_4^{(k)} = 0 \quad \text{on} \quad S,$$

$$[TW_{A,s}(g^{(k)})]_{\Sigma}^+ + [TW_{A,\Sigma}(g_4^{(k)})n]_{\Sigma}^+ + \mathcal{L}_{A,s}\psi^{(k)} = 0 \quad \text{on} \quad \Sigma,$$

$$[W_{A,s}(g^{(k)})]_{\Sigma}^+ + [W_{A,\Sigma}(g_4^{(k)})n]_{\Sigma}^+ + \mathcal{K}_{A,s}^2\psi^{(k)} + \varphi^{(k)} = 0 \quad \text{on} \quad \Sigma.$$

Taking into consideration that $g_4^{(k)} = 0$ and $g^{(k)} \cdot n = 0$ on $S$, there remains to verify the following equalities

$$-b_2^2D^\top_{a,s}(g^{(k)} \cdot n) = 0 \quad \text{on} \quad S,$$

$$[TW_{A,s}(g^{(k)})]_{\Sigma} + \mathcal{L}_{A,s}\psi^{(k)} = 0 \quad \text{on} \quad \Sigma,$$

$$[W_{A,s}(g^{(k)})]_{\Sigma} + \mathcal{K}_{A,s}^2\psi^{(k)} + \varphi^{(k)} = 0 \quad \text{on} \quad \Sigma.$$
These relations directly follow from the representation (3.26) and conditions (3.25). This proves that the vector $G^{(k)}$ belongs to $\ker P^T$.

Furthermore, we show that $\dim \ker P^T = \dim X_\omega (\Omega^+)$, and that the relations (3.24), (3.27), and (3.28) actually arrange a one-to-one correspondence between the linearly independent Jones modes and linearly independent elements of $\ker P^T$. In fact, applying these relations together with (3.26) we conclude that to linearly independent elements $\{G^{(j)}\}_{p=1}^P (1 \leq p \leq \ell = \dim \ker P^T)$ of $\ker P^T$ there correspond linearly independent Jones modes $\{v^{(k)}\}_{p=1}^P$, i.e., $\dim \ker P^T \leq \dim X_\omega (\Omega^+)$. We can show the inverse inequality in the same way, which proves the lemma.

Now, we can write the necessary and sufficient conditions for the solvability of the system (3.12) as the following equality $\langle Q, G^* \rangle \equiv \langle F, g^* \rangle_\Sigma + 2^{-1} \langle F^+ - F^-, \psi^* \rangle_\Sigma + \langle F^- - F^+, \varphi^* \rangle_\Sigma = 0$ for all $G^* \in \ker P^T$.

Due to the pointwise orthogonality of the vector $g^*$ to the normal $n$ and since $F$ is parallel to $n$ in accordance with (2.6), the first term in the last equality vanishes. Therefore, in view of Lemma 3.2, the above formulated necessary and sufficient conditions for our transmission problem can be written as

$$\langle F^+ + F^-, [u^*]^+ - [u^*]^-, \rangle_\Sigma + \langle F^+ - F^-, [u^*]^+ + [u^*]^-, \rangle_\Sigma = 0,$$

where $u^*$ is an arbitrary Jones mode, $u^* \in X_\omega (\Omega^+)$. Clearly, for the traction free crack conditions, i.e., when $F^+ = F^- = 0$ on $\Sigma$ then the necessary and sufficient conditions are automatically satisfied. Thus, we have proved the following existence results for our original transmission problem (see (2.1), (2.2), (2.4), (2.5), and (2.8)).

**Theorem 3.3** The nonhomogenous problem (TP) with boundary and transmission data as in (2.6) and (2.9) is solvable if and only if the condition (3.29) is fulfilled. The corresponding elastic vector field of displacements $u \in \left[ H^{2/3}_1 (\Omega^+\Sigma) \right]^3$ and the radiating scalar field $w^{(sc)} \in H^{1}_{1,\text{loc}} (\Omega^-) \cap \text{Som} (\Omega^-)$ are representable in the form (3.6)–(3.7) where the density functions are defined by the linear system of integral equations (3.8)–(3.11). The scattered field $w^{(sc)}$ is defined uniquely, while the elastic displacement field is defined modulo Jones mode.

**Corollary 3.4** For the traction free crack conditions, i.e., when $F^+ = F^- = 0$ on $\Sigma$, the transmission problem (TP) is solvable for arbitrary frequency parameter $\omega$ and for arbitrary incident field function $u^{(inc)}$. Again the scattered field $w^{(sc)}$ is defined uniquely, while the elastic displacement field is defined modulo Jones mode.

### 3.3 Regularity results

Throughout this subsection $L_p, \ W^r_p, \ H^s_p, \ \text{and} \ B^{2r}_{p,q}$ (with $r \geq 0$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$) denote the well-known Lebesgue, Sobolev–Slobodetskii, Bessel potential, and Besov function spaces, respectively. We recall that $H^s_p = B^s_p$, $H^s_p = B^s_p$, $W^r_p = B^r_{p,p}$, and $H^s_p = W^s_p$, for any $r \geq 0$, $s \in \mathbb{R}$, for any positive and non-integer $r$, and for any nonnegative integer $k$ (see, e.g., [32] and [21]).

Here we establish the almost best possible regularity results for the solutions of the transmission problem (TP). In particular, with the help of embedding theorems we prove the following assertions.

**Theorem 3.5** Let the boundary data of the problem involved in the conditions (2.4), (2.5), and (2.8) possess the property (3.29), as well as (2.6) and (2.9), within the new space framework

$$f_0 \in B^{-\frac{1}{2}}_{p,p} (S), \quad F = (F_1, F_2, F_3)^T \in \left[ B^{-\frac{1}{2}}_{p,p} (S) \right]^3,$$

$$r_\Sigma F = r_{\Sigma} (F_1, F_2, F_3)^T \in \left[ B^{-\frac{1}{2}}_{p,p} (\Sigma) \right]^3, \quad F^+ - F^- \in \left[ B^{-\frac{1}{2}}_{p,p} (\Sigma) \right]^3,$$

with $4/3 < p < 4$.

Then for a solution pair $(u, w^{(sc)})$ of the transmission problem (TP) the following embedding holds

$$(u, w^{(sc)}) \in \left[ H^{1/2}_p (\Omega^+\Sigma) \right]^3 \times \left[ H^{1}_{1,\text{loc}} (\Omega^-) \cap \text{Som} (\Omega^-) \right].$$
Proof. Evidently, by the representations (3.6)–(3.7) we arrive again at the system of integral equations (3.8)–(3.11) or (3.12).

First of all we note that the operators

\[-2^{-1}I_3 + K_{A,s}^{(1)} - iH_{A,s} : [H_p^s(S)]^3 \rightarrow [H_p^s(S)]^3, \quad N_{s,s} : H_p^s(S) \rightarrow H_p^{s-1}(S),\]

are invertible for arbitrary \(s \in \mathbb{R}\) and arbitrary \(1 < p < \infty\), while the operator

\[L_{A,s} : \tilde{H}_p^s(S) \rightarrow \tilde{H}_p^{s-1}(S)\]

is invertible if the condition \(1/p - 1/2 < s < 1/p + 1/2\) holds (cf. [Appendix, Theorems A.2, A.3, and A.5]). This inequality with \(s = 1 - 1/p\) gives the following restriction for \(p : 4/3 < p < 4\).

Further, let

\[X_p := \left[\tilde{H}_p^\frac{1}{3} (S)\right]^3 \times H_p^{1-\frac{1}{3}} (S) \times \left[\tilde{H}_p^{1-\frac{1}{3}} (\Sigma)\right]^3 \times \left[\tilde{H}_p^{-\frac{1}{3}} (\Sigma)\right]^3,\]

\[Y_p := \left[\tilde{H}_p^\frac{1}{3} (S)\right]^3 \times H_p^{1-\frac{1}{3}} (S) \times \left[\tilde{H}_p^{-\frac{1}{3}} (\Sigma)\right]^3 \times r_s \tilde{H}_p^{-\frac{1}{3}} (\Sigma)^3.\]

Now, applying word-by-word the arguments used in the proof of Theorem 3.1 we obtain that the operator

\[\mathcal{P} : X_p \rightarrow Y_p \quad \text{for} \quad 4/3 < p < 4 \quad (3.31)\]

is Fredholm with zero index.

The null space of the operator (3.31) coincides with the null space of the operator (3.13) due to Theorems A.2 and A.4. The same is true for the corresponding adjoint operators. Therefore, the necessary and sufficient conditions for the solvability of the equation

\[\mathcal{P}G = Q \quad (3.32)\]

with a vector \(G := (g, g_4, \psi, \varphi)^T \in X_p\) and a given right-hand side vector \(Q = (F, f_0, (F^+ + F^-)/2, F^+ - F^-)^T \in Y_p\), can again be expressed as (3.29), where the brackets denote the duality between the spaces \([B_p^s(\Sigma)]^3\) and \([\tilde{B}_p^s(\Sigma)]^3\). Since these conditions are satisfied, in accordance with the assumptions of the theorem, we conclude that (3.32) is solvable. Now, the representation formulae (3.6)–(3.7) and Theorem A.2 (see Appendix) completes the proof.

**Theorem 3.6** Let \(f_0, F, F^\pm, (u, w^{(sc)})\) be as in Theorem 3.5, and let the conditions

\[4/3 < p < 4, \quad 1 < t < \infty, \quad 1 \leq q \leq \infty, \quad 1/t - 1/2 < s < 1/t + 1/2,\]

be fulfilled.

In addition to (3.30),

(i) if \(f_0 \in B^t_{s,t}(S), \quad F = (F_1, F_2, F_3)^T \in [B^t_{s,t}(S)]^3, \quad r_{\Sigma} F^\pm = r_{\Sigma}(F_{1}^{\pm}, F_{2}^{\pm}, F_{3}^{\pm})^T \in [B^t_{s,t}(\Sigma)]^3, \quad F^+ - F^- \in [B^t_{s,t}(\Sigma)]^3, \quad \text{then}\]

\[\left(\begin{array}{c}
(u, w^{(sc)})
\end{array}\right) \in \left[H_{t,1}^{1+\frac{1}{t}} (\Omega_{1}^{\ast})\right]^3 \times \left[H_{t,1,loc}^{1+\frac{1}{t}} (\Omega_{t,1}^{\ast}) \cap \text{Som}(\Omega_{t,1}^{\ast})\right];\]

(ii) if \(f_0 \in B_{s,q} (S), \quad F = (F_1, F_2, F_3)^T \in [B^t_{s,q}(S)]^3, \quad r_{\Sigma} F^\pm = r_{\Sigma}(F_{1}^{\pm}, F_{2}^{\pm}, F_{3}^{\pm})^T \in [B^t_{s,q}(\Sigma)]^3, \quad F^+ - F^- \in [B^t_{s,q}(\Sigma)]^3, \quad \text{then}\]

\[\left(\begin{array}{c}
(u, w^{(sc)})
\end{array}\right) \in \left[B_{t,q}^{1+\frac{1}{t}} (\Omega_{1}^{\ast})\right]^3 \times \left[B_{t,q,loc}^{1+\frac{1}{t}} (\Omega_{t,1}^{\ast}) \cap \text{Som}(\Omega_{t,1}^{\ast})\right];\]
\(\partial\) is the distance from a reference point \(T_u\) of the corresponding stress vector \(\sigma_t\).

Clearly, \(\kappa = \min\{1, 2/3\}\). Since \(t\) is sufficiently large and \(\varepsilon\) is sufficiently small, the embedding (3.34) completes the proof.

We point out that the embedding (3.33) gives the almost best possible regularity result for the elastic displacement field in a neighbourhood of the edge \(\partial\Sigma\). Note that, even for \(C^\infty\)-regular boundary data the displacement field does not possess \(C^t\)-smoothness with \(t > 1/2\) in a neighbourhood of \(\partial\Sigma\), in general. Actually, the regularity results obtained lead to the following corollary: \(the\ dominant\ stress\ singularity\ exponent\ equals\ to\ -1/2\). In fact, more detailed analysis based on the asymptotic expansions of solutions (see \([5, 7]\) and \([10]\)) shows that for sufficiently smooth boundary data, e.g., \(C^\infty\)-smooth data say, the principal singular terms of the solution vector \(u\) near the curve \(\partial\Sigma\) can be represented by a product of a “good” vector-function and a factor \([\varrho(x)]^{1/2}\). Here \(\varrho(x)\) is the distance from a reference point \(x\) to the curve \(\partial\Sigma\). Therefore, near the edge \(\partial\Sigma\) the principal singular term of the corresponding stress vector \(T\sigma\) is represented as a product of a “good” vector-function (the so-called stress intensity factor) and the factor \([\varrho(x)]^{-1/2}\). We only notice here that the exponent of this factor (i.e., the dominant stress singularity exponent) is determined uniquely by the eigenvalues of the special auxiliary matrix constructed by using the principal homogeneous symbol matrix of the operator \(L_{A,V}\). We also remark that the entries of this principal homogeneous symbol matrix \(\sigma(L_{A,V})(\xi; z), \xi \in \mathbb{R}^2, z \in \partial\Sigma,\) are even functions in \(\xi \in \mathbb{R}^2\) (see, e.g., \([11]\) and \([24]\)), and therefore the special auxiliary matrix mentioned above, which has the form

\[
\left[\sigma(L_{A,V})(0, +1; z)\right]^{-1}\left[\sigma(L_{A,V})(0, -1; z)\right], \quad z \in \partial\Sigma,
\]

actually is the unit \(3 \times 3\) matrix. Consequently, the corresponding eigenvalues are \(\lambda_j = 1, j = 1, 2, 3\). The stress singularity exponents \(\varkappa_j = a_j + i b_j\) corresponding to the eigenvalues \(\lambda_j\) are calculated by the formulae (cf. \([5]\) and \([7]\))

\[
a_j = -\frac{1}{2} + \frac{\arg \lambda_j}{2\pi}, \quad b_j = -\frac{\ln |\lambda_j|}{2\pi}, \quad j = 1, 2, 3.
\]

Clearly, \(\varkappa_0 = \min\{a_1, a_2, a_3\}\) is then the dominant stress singularity exponent.

The above formulae show that in our case \(\varkappa_j = a_j = -1/2\), since \(\arg \lambda_j = 0\) and \(\ln |\lambda_j| = 0, j = 1, 2, 3\). Thus, the dominant stress singularity exponent is \(\varkappa_0 = -1/2\).

On the other hand it is evident that the elastic displacement field and the exterior scattered field in the corresponding one-sided closed neighbourhoods of the surface \(S\) may possess higher regularity. To exemplify such evidence, let us consider the Hölder spaces \(C^{m,\alpha}\) (with \(m\) being a nonnegative integer and \(0 < \alpha \leq 1\)), i.e.,
the \( m \) times continuous differentiable elements whose \( m \)-th order derivatives are Hölder continuous with exponent \( \nu \). From the structure of the system of integral equations (3.8)–(3.11) it follows that if \( S \in C^{k+2, \alpha'} \), \( F_j, f_j \in C^{k, \alpha}(S), j = 1, 2, 3 \), where \( k \geq 0 \) is an integer and \( 0 < \alpha < \alpha' \leq 1 \), then \( g_j \in C^{k, \alpha}, j = 1, 2, 3 \), and \( g_4 \in C^{k+1, \alpha} \). By Theorem A.1 this yields that \( u \) and \( u^{(\infty)} \) represented by (3.6)–(3.7) are \( C^{k+1, \alpha} \)-regular in the one-sided closed neighbourhoods of the surface \( S \).

A Appendix

A.1 Sommerfeld–Kupradze radiation conditions

We say that a vector \( u = (u_1, u_2, u_3)^T \) satisfies the Sommerfeld–Kupradze radiation conditions in \( \Omega^- \) and write \( u \in \mathcal{S}(\Omega^-) \) if

\[
u(x) = u^{(1)}(x) + u^{(2)}(x) \quad \text{in} \quad \Omega^-,
\]

where \( u^{(l)} = (u_1^{(l)}, u_2^{(l)}, u_3^{(l)})^T \) are metaharmonic vectors, \( [\Delta + k_2^2]u^{(l)}(x) = 0 \) (\( l = 1, 2 \)), with \( k_1 \) and \( k_2 \) given by (3.2), and

\[
\frac{\partial}{\partial |x|} u_p^{(l)}(x) - ik_l u_p^{(l)}(x) = \mathcal{O}(|x|^{-2}) \quad \text{as} \quad |x| \to \infty, \quad p = 1, 2, 3.
\]

The homogeneous exterior Dirichlet and Neumann type boundary value problems for the steady state elastic oscillation equations in the class of vectors satisfying the Sommerfeld–Kupradze radiation conditions in \( \Omega^- \)

\[
A(\partial, \omega) u = 0 \quad \text{in} \quad \Omega^-, \quad u \in [H^1_{2, \text{loc}}(\Omega^-)]^3 \cap \mathcal{S}(\Omega^-),
\]

\[
[u]^S = 0 \quad \text{or} \quad [T(\partial, n)u]^S = 0
\]

possess only the trivial solution (for details see [20]).

A.2 Properties of potentials and boundary operators

The jump and mapping properties of the single and double layer potentials and the corresponding boundary integral (pseudodifferential) operators in the Hölder (\( C^{k, \alpha} \)), Sobolev–Slobodetskii (\( W^s_p \)), Bessel potential (\( H^s_p \)) and Besov (\( B^s_p \)) spaces are well studied (see, e.g., [11]–[14], [20], [23]–[26]). We will employ the notation introduced in Subsection 3.1 and formulate these properties in the form of theorems.

**Theorem A.1** Let \( S \in C^{k+1, \alpha'} \) for some integer \( k \geq 0 \) and \( 0 < \alpha' \leq 1 \), and consider \( 0 < \alpha < \alpha' \). Then the operators

\[
V_{A,S} : [C^{k, \alpha}(S)]^3 \longrightarrow [C^{k+1, \alpha}(\Omega^+)]^3,
\]

\[
W_{A,S} : [C^{k, \alpha}(S)]^3 \longrightarrow [C^{k, \alpha}(\Omega^+)]^3
\]

are bounded.

For any \( g \in [C^{k, \alpha}(S)]^3 \) and any \( x \in S \)

\[
[V_{A,S}(g)(x)]^\pm = V_{A,S}(g)(x) = \mathcal{H}_{A,S} g(x), \quad k \geq 0,
\]

\[
[T(\partial_x, n(x))V_{A,S}(g)(x)]^\pm = \pm 2^{-1}I_3 + K^{(1)}_{A,S} g(x), \quad k \geq 0,
\]

\[
[W_{A,S}(g)(x)]^\pm = \pm 2^{-1}I_3 + K^{(2)}_{A,S} g(x), \quad k \geq 0,
\]

\[
[T(\partial_x, n(x))W_{A,S}(g)(x)]^- = [T(\partial_x, n(x))W_{A,S}(g)(x)]^- = \mathcal{L}_{A,S} g(x), \quad k \geq 1,
\]

where \( \mathcal{H}_{A,S}, K^{(1)}_{A,S} \), and \( K^{(2)}_{A,S} \) are the integral operators given by (3.3)–(3.5).

For \( S \in C^\infty \) the operators \( \mathcal{H}_{A,S}, \pm 2^{-1}I_3 + K^{(1)}_{A,S}, \pm 2^{-1}I_3 + K^{(2)}_{A,S} \), and \( \mathcal{L}_{A,S} \) are elliptic pseudodifferential operators of order \(-1, 0, 0, \) and \(1\), respectively.
Moreover, the principal homogeneous symbol matrices of the singular integral operators $\pm 2^{-1} I_3 + K^{(1)}_{A,S}$ and $\pm 2^{-1} I_3 + K^{(2)}_{A,S}$ are nondegenerate, while the principal homogeneous symbol matrices of the weakly singular operator $H_{A,S}$ and the singular integro-differential operator $L_{A,S}$ have the following mapping properties

\begin{align}
H_{A,S} & : \left[ C^{k,\alpha}(S) \right]^3 \longrightarrow \left[ C^{k+1,\alpha}(S) \right]^3, \\
K^{(1)}_{A,S} & : \left[ C^{k,\alpha}(S) \right]^3 \longrightarrow \left[ C^{k,\alpha}(S) \right]^3, \\
K^{(2)}_{A,S} & : \left[ C^{k,\alpha}(S) \right]^3 \longrightarrow \left[ C^{k,\alpha}(S) \right]^3, \\
L_{A,S} & : \left[ C^{k+1,\alpha}(S) \right]^3 \longrightarrow \left[ C^{k,\alpha}(S) \right]^3.
\end{align}

**Theorem A.2** Let $V_{A,S}$, $W_{A,S}$, $H_{A,S}$, $K^{(1)}_{A,S}$, $K^{(2)}_{A,S}$, and $L_{A,S}$ be as in Theorem A.1 and consider $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, and $S \in C^\infty$. The operators in (A.1) and (A.6) can be extended continuously to the following bounded operators

\begin{align}
V_{A,S} & : \left[ H^{2,\frac{1}{2}}_2(S) \right]^3 \longrightarrow \left[ H^{2,\frac{1}{2}}_2(\Omega^+) \right]^3, \\
& \quad : \left[ B^{s}_{p,p}(S) \right]^3 \longrightarrow \left[ H^{s+1,\frac{1}{2}}_p(\Omega^+) \right]^3, \\
& \quad : \left[ B^{s}_{p,q}(S) \right]^3 \longrightarrow \left[ B^{s+1,\frac{1}{2}}_{p,q}(\Omega^+) \right]^3, \\
W_{A,S} & : \left[ H^{1,\frac{1}{2}}_2(S) \right]^3 \longrightarrow \left[ H^{1,\frac{1}{2}}_2(\Omega^+) \right]^3, \\
& \quad : \left[ B^{s}_{p,p}(S) \right]^3 \longrightarrow \left[ H^{s+\frac{1}{2}}_p(\Omega^+) \right]^3, \\
& \quad : \left[ B^{s}_{p,q}(S) \right]^3 \longrightarrow \left[ B^{s+\frac{1}{2}}_{p,q}(\Omega^+) \right]^3, \\
H_{A,S} & : \left[ H^{2,\frac{1}{2}}_2(S) \right]^3 \longrightarrow \left[ H^{2,\frac{1}{2}}_2(S) \right]^3, \\
& \quad : \left[ H^{s}_{p}(S) \right]^3 \longrightarrow \left[ H^{s+1}_{p}(S) \right]^3, \\
& \quad : \left[ B^{s}_{p,q}(S) \right]^3 \longrightarrow \left[ B^{s+1}_{p,q}(S) \right]^3, \\
K^{(1)}_{A,S} & : \left[ H^{2,\frac{1}{2}}_2(S) \right]^3 \longrightarrow \left[ H^{2,\frac{1}{2}}_2(S) \right]^3, \\
& \quad : \left[ H^{s}_{p}(S) \right]^3 \longrightarrow \left[ H^{s}_{p}(S) \right]^3, \\
& \quad : \left[ B^{s}_{p,q}(S) \right]^3 \longrightarrow \left[ B^{s}_{p,q}(S) \right]^3, \\
K^{(2)}_{A,S} & : \left[ H^{2,\frac{1}{2}}_2(S) \right]^3 \longrightarrow \left[ H^{2,\frac{1}{2}}_2(S) \right]^3, \\
& \quad : \left[ B^{s}_{p,q}(S) \right]^3 \longrightarrow \left[ B^{s}_{p,q}(S) \right]^3, \\
L_{A,S} & : \left[ H^{2,\frac{1}{2}}_2(S) \right]^3 \longrightarrow \left[ H^{2,\frac{1}{2}}_2(S) \right]^3, \\
& \quad : \left[ B^{s}_{p,q}(S) \right]^3 \longrightarrow \left[ B^{s}_{p,q}(S) \right]^3.
\end{align}

The relations (A.2)–(A.5) remain valid in the corresponding just presented function spaces.
The operators \( \mathcal{H}_{A,S} \), \( ± 2^{-1} I_3 + K_{A,S}^{(1)} \), \( ± 2^{-1} I_3 + K_{A,S}^{(2)} \), and \( \mathcal{L}_{A,S} \) are Fredholm with zero index and their null-spaces are invariant with respect to \( p, q, \) and \( s \).

The operators

\[
-2^{-1} I_3 + K_{A,S}^{(1)} - i \mathcal{H}_{A,S} : \left[ H_2^{\alpha_1}(S) \right]^3 \rightarrow \left[ H_2^{\alpha_2}(S) \right]^3,
\]

\[
: \left[ H_p^s(S) \right]^3 \rightarrow \left[ H_p^s(S) \right]^3,
\]

\[
: \left[ B_{p,q}^s(S) \right]^3 \rightarrow \left[ B_{p,q}^s(S) \right]^3,
\]

are invertible.

Similar properties also hold for the scalar single and double layer potentials, \( V_{A,S} \) and \( W_{A,S} \), and for the corresponding boundary integral operators \( \mathcal{H}_{A,S}, K_{A,S}^{(1)}, K_{A,S}^{(2)}, \) and \( \mathcal{L}_{A,S} \). Note that \( \mathcal{H}_{A,S} \) is an integral operator with weakly singular kernel. If \( S \in C^{k+1,\alpha'} \) for some integer \( k \geq 0 \) and \( 0 < \alpha' \leq 1 \), then \( K_{A,S}^{(1)} \) and \( K_{A,S}^{(2)} \) are also integral operators with weakly singular kernels, while \( \mathcal{L}_{A,S} \) is a singular integro-differential operator (for \( k \geq 1 \)) [26].

Theorem A.3 Let \( k, \alpha, \alpha', p, q, \) and \( S \) be as either in Theorem A.1 or in Theorem A.2. Then the operators

\[
\mathcal{D}_{A,S} : C^{k+1,\alpha}(S) \rightarrow C^{k+1,\alpha}(S),
\]

\[
: H_p^s(S) \rightarrow H_p^s(S),
\]

\[
: B_{p,q}^s(S) \rightarrow B_{p,q}^s(S),
\]

\[
\mathcal{N}_{A,S} : C^{k+1,\alpha}(S) \rightarrow C^{k,\alpha}(S),
\]

\[
: H_p^s(S) \rightarrow H_p^{s-1}(S),
\]

\[
: B_{p,q}^s(S) \rightarrow B_{p,q}^{s-1}(S)
\]

are invertible. Here \( \mathcal{D}_{A,S} := -2^{-1} I + K_{A,S}^{(2)} - i \mathcal{H}_{A,S} \) and \( \mathcal{N}_{A,S} := \mathcal{L}_{A,S} - i \left[ 2^{-1} I + K_{A,S}^{(1)} \right] \).

A.3 Some results for pseudodifferential equations on manifolds with boundary

Here we shall present some results from the theory of strongly elliptic pseudodifferential equations on manifolds with boundary in Bessel potential and Besov spaces which are the main tools for the analysis of crack type problems by the potential method. In particular, in our investigation we need some results describing the Fredholm properties of pseudodifferential operators on a compact manifold with boundary. Their complete description can be found in [8], [14], [16] and [31].

Let \( \overline{M} \subset C^\infty \) be a compact, \( n \)-dimensional, nonselfintersecting manifold with boundary \( \partial \overline{M} \subset C^\infty \) and let \( A \) be a strongly elliptic \( N \times N \) matrix pseudodifferential operator of order \( \nu \in \mathbb{R} \) on \( \overline{M} \). Denote by \( \sigma_A(x, \xi) \) the principal homogeneous symbol of the operator \( A \) in some local coordinate system \( \{ x \in \overline{M}, \xi \in \mathbb{R}^n \setminus \{ 0 \} \} \).

Let \( \lambda_1(x), \ldots, \lambda_N(x) \) be the eigenvalues of the matrix

\[
[ \sigma_A(x, 0, 0, \ldots, 0, +1) ]^{-1} \sigma_A(x, 0, 0, \ldots, 0, -1), \quad x \in \partial \overline{M},
\]

and introduce the notation \( \delta_j(x) = \Re \left[ (2\pi i)^{-1} \ln \lambda_j(x) \right] \), \( j = 1, \ldots, N \). Here the branch in the logarithmic function \( \ln \tau \) is chosen with regard to the inequality \( -\pi < \arg \tau \leq \pi \), \( j = 1, \ldots, N \). Due to the strong ellipticity of \( A \) we have the strong inequality \( -1/2 < \delta_j(x) < 1/2 \) for \( x \in \partial \overline{M} \).

Note that the numbers \( \delta_j(x) \) do not depend on the choice of the local coordinate system.

Observe also that in the particular case when \( \sigma_A(x, \xi) \) is a positive definite matrix for every \( x \in \overline{M} \) and \( \xi \in \mathbb{R}^n \setminus \{ 0 \} \) we have \( \delta_j(x) = 0 \) for \( j = 1, \ldots, N \), since all the eigenvalues \( \lambda_j(x) \) \( (j = 1, \ldots, N) \) are positive numbers for any \( x \in \overline{M} \).

It is also evident that if the entries of the principal symbol matrix \( \sigma_A(x, \xi) \) are even functions in \( \xi \) for any \( x \in \overline{M} \), then \( \lambda_j(x) = 1 \) and again \( \delta_j(x) = 0 \) for \( j = 1, \ldots, N \).

The Fredholm properties of strongly elliptic pseudodifferential operators are characterized in the following theorem.

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Theorem A.4 Let $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$, and let $A$ be a strongly elliptic pseudodifferential operator of order $\nu$ in $\mathbb{R}^n$, i.e., there is a positive constant $c_0$ such that $\Re \sigma_A(x, \xi) \zeta \cdot \zeta \geq c_0 |\xi|^2$ for $x \in \mathbb{R}^n$, $\xi \in \mathbb{R}^n$ with $|\xi| = 1$, and $\zeta \in \mathbb{C}^N$. Then

$$A : \left[ H^s_p(M) \right] \rightarrow \left[ H^s_p \nu(M) \right], \quad (A.7)$$

$$B_{p,q}^{s-\nu}(M) \rightarrow \left[ B_{p,q}^{s-\nu}(M) \right], \quad (A.8)$$

are Fredholm operators with index zero if

$$\frac{1}{p} - 1 + \sup_{x \in \partial M, 1 \leq j \leq N} \delta_j(x) < s - \frac{\nu}{2} < \frac{1}{p} + \inf_{x \in \partial M, 1 \leq j \leq N} \delta_j(x). \quad (A.9)$$

Moreover, the null-spaces and indices of the operators $A$ in (A.7) and (A.8) are the same (for all values of the parameter $q \in [1, +\infty]$) provided $p$ and $s$ satisfy the inequality (A.9).

In particular, the following assertion holds [11], [12] and [24].

Theorem A.5 The operators $L_{A, \Sigma}$, $\left[ H^s_p(\Sigma) \right] \rightarrow \left[ H^s_p \nu(\Sigma) \right]$ and $L_{A, \Sigma}$, $\left[ \tilde{B}_{p,q}^{s-\nu}(\Sigma) \right] \rightarrow \left[ B_{p,q}^{s-\nu}(\Sigma) \right]$ are invertible if $1/p - 1/2 < s < 1/p + 1/2$.

References