Mathematical problems of the theory of elasticity of chiral materials for Lipschitz domains

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Communicated by M. Mitrea

SUMMARY
By the potential method, we investigate the Dirichlet and Neumann boundary value problems of the elasticity theory of hemitropic (chiral) materials in the case of Lipschitz domains. We study properties of the single- and double-layer potentials and of certain, generated by them, boundary integral operators. These results are applied to reduce the boundary value problems to the equivalent first and the second kind integral equations and the uniqueness and existence theorems are proved in various function spaces. Copyright © 2005 John Wiley & Sons, Ltd.

KEY WORDS: elasticity theory; elastic chiral materials; potential theory; boundary value problems

Dedicated to the memory of Professor Victor Kupradze on the occasion of the 100th birth anniversary

1. INTRODUCTION
A solid which is not isotropic with respect to inversion is called non-centrosymmetric, acentric, hemitropic, or chiral. Materials may exhibit chirality on the atomic scale, as in quartz and in biological molecules, as well as on a large scale, as in composites with helical or screw-shaped inclusions (for details see, e.g. References [1,2] and the references therein).

In recent years the electromagnetic field in chiral media has been the object of intensive research, see e.g. References [3–6], and the references therein.

Mathematical models describing the chiral properties of elastic materials have been proposed by Aero and Kuvshinski [1,7] (for the history of the problem see also References [8–11] and the references therein).

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Received 18 November 2004
Revised 2 December 2004
Accepted 10 January 2005

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Particular problems of the elasticity theory of hemitropic continuum related to the present paper have been considered in References [2,9,11–17].

In Reference [18] the basic boundary value problems (BVP) of the non-centrosymmetric theory of elasticity for bodies with smooth boundaries are studied by the classical potential method and the existence and uniqueness results are obtained with the help of the boundary integral equations method in various function spaces. In the present paper, we treat the same BVPs for Lipschitz domains.

Over the past two decades, there has been considerable progress in treating boundary value problems by the layer potential method in Lipschitz domains for scalar elliptic equations, e.g. Laplace’s equation, and for some special elliptic systems such as the Stokes system, Lamé equations of the classical elastostatics (isotropic case) and Maxwell equations (see, e.g. References [19–28], and the references therein). However, it should be mentioned that in the elasticity theory of anisotropic bodies the invertibility of the second kind operators corresponding to the Dirichlet and the Neumann BVPs is still an open question.

The main goal of our investigation is to study (by the layer potential method) the three-dimensional Dirichlet and Neumann BVPs of the theory of elasticity of hemitropic bodies with Lipschitz boundaries. Here essential difficulties arise in the study of invertibility of the corresponding, strongly singular, boundary integral operators with non-smooth symbols. In general, these operators are not classical pseudodifferential operators due to the Lipschitz smoothness of the boundary. Therefore, in this case one needs a special approach which is completely different from the case of smooth domains.

The paper is organized as follows.

In Section 2, we give an overview concerning the basic mechanical characteristics of the theory of elasticity of chiral materials and treat the mathematical aspects related to the weak formulation of the corresponding BVPs.

In Sections 3 and 4, we introduce the fundamental matrices of solutions, construct the generalized layer and Newtonian potentials, and derive the general integral representation formulae of solutions to the system of differential equations of hemitropic elasticity.

Section 5 is devoted to the analysis of properties of the layer potentials.

In Section 6, we establish the Rellich–Payne–Weinbereger-type identity and derive some standard auxiliary estimates.

Finally, in Section 7 we prove basic existence and regularity results for the Dirichlet and Neumann type BVPs.

2. BASIC EQUATIONS AND GREEN FORMULAE

2.1. Constitutive equations

Let \( \mathbb{R}^3 \) be the three-dimensional Euclidean space and \( \Omega^+ \subset \mathbb{R}^3 \) be a bounded Lipschitz domain with a boundary \( S := \partial \Omega^+ \), \( \Omega^+ = \Omega \cup S \); \( \Omega^- = \mathbb{R}^3 \setminus \Omega^+ \). We assume that \( \Omega \in \{ \Omega^+, \Omega^- \} \) is filled with an elastic material possessing the hemitropic properties.

Denote by \( u = (u_1, u_2, u_3)^\top \) and \( \omega = (\omega_1, \omega_2, \omega_3)^\top \) the displacement vector and the micro-rotation vector, respectively; here and in what follows the symbol \((\cdot)^\top\) denotes transposition. Note that the micro-rotation vector in the hemitropic elasticity theory is kinematically distinct from the macro-rotation vector \( \frac{1}{2} \text{curl} u \).
The tensors of the force stress \(\{\tau_{pq}\}\) and the couple stress \(\{\mu_{pq}\}\) in the linear theory are as follows (the constitutive equations):

\[
\tau_{pq} = \tau_{pq}(U) := (\mu + \nu) \frac{\partial u_q}{\partial x_p} + (\mu - \nu) \frac{\partial u_p}{\partial x_q} + \lambda \delta_{pq} \text{div } u + \delta \delta_{pq} \text{div } \omega \\
+ (\kappa + \nu) \frac{\partial \omega_q}{\partial x_p} + (\kappa - \nu) \frac{\partial \omega_p}{\partial x_q} - 2\alpha \sum_{k=1}^{3} \varepsilon_{pqk} \omega_k
\]

\[
\mu_{pq} = \mu_{pq}(U) := \delta \delta_{pq} \text{div } u + (\kappa + \nu) \left[ \frac{\partial u_p}{\partial x_q} - \frac{3}{2} \sum_{k=1}^{3} \varepsilon_{pqk} \omega_k \right] + \beta \delta \delta_{pq} \text{div } \omega \\
+ (\kappa - \nu) \left[ \frac{\partial \omega_p}{\partial x_q} - \sum_{k=1}^{3} \varepsilon_{qpk} \omega_k \right] + (\gamma + \varepsilon) \frac{\partial \omega_q}{\partial x_p} + (\gamma - \varepsilon) \frac{\partial \omega_p}{\partial x_q}
\]

where \(U = (u, \omega)^T\), \(\delta_{pq}\) is the Kronecker delta, \(\varepsilon_{pqk}\) is the permutation (Levi–Civitá) symbol, and \(\alpha, \beta, \gamma, \delta, \lambda, \mu, \nu, \kappa, \) and \(\varepsilon\) are the material constants (see Reference [1]).

The components of the vectors of the force stress \(\tau^{(n)}\) and the coupled stress \(\mu^{(n)}\), acting on a surface element with a normal vector \(n = (n_1, n_2, n_3)\), read as

\[
\tau_q^{(n)} = \sum_{p=1}^{3} \tau_{pq} n_p, \quad \mu_q^{(n)} = \sum_{p=1}^{3} \mu_{pq} n_p, \quad q = 1, 2, 3
\]

Let us introduce the generalized stress operator (6 \(\times\) 6 matrix differential operator)

\[
T(\hat{\partial}, n) = \begin{bmatrix}
T^{(1)}(\hat{\partial}, n) & T^{(2)}(\hat{\partial}, n) \\
T^{(3)}(\hat{\partial}, n) & T^{(4)}(\hat{\partial}, n)
\end{bmatrix}_{6 \times 6}, \quad T^{(j)} = [T_{pq}^{(j)}]_{3 \times 3}, \quad j = 1, 4
\]

where \(\hat{\partial} = (\hat{\partial}_1, \hat{\partial}_2, \hat{\partial}_3), \quad \hat{\partial}_j = \hat{\partial}/\partial x_j\)

\[
T^{(1)}_{pq}(\hat{\partial}, n) = (\mu + \nu) \hat{\partial}_{pq} \frac{\partial}{\partial n} + (\mu - \nu) n_q \frac{\partial}{\partial x_p} + \lambda n_p \frac{\partial}{\partial x_q}
\]

\[
T^{(2)}_{pq}(\hat{\partial}, n) = (\kappa + \nu) \hat{\partial}_{pq} \frac{\partial}{\partial n} + (\kappa - \nu) n_q \frac{\partial}{\partial x_p} + \delta \delta_{pq} \frac{\partial}{\partial x_q} - 2\alpha \sum_{k=1}^{3} \varepsilon_{pqk} n_k
\]

\[
T^{(3)}_{pq}(\hat{\partial}, n) = (\kappa + \nu) \hat{\partial}_{pq} \frac{\partial}{\partial n} + (\kappa - \nu) n_q \frac{\partial}{\partial x_p} + \delta \delta_{pq} \frac{\partial}{\partial x_q}
\]

\[
T^{(4)}_{pq}(\hat{\partial}, n) = (\gamma + \varepsilon) \hat{\partial}_{pq} \frac{\partial}{\partial n} + (\gamma - \varepsilon) n_q \frac{\partial}{\partial x_p} + \beta n_p \frac{\partial}{\partial x_q} - 2\nu \sum_{k=1}^{3} \varepsilon_{pqk} n_k
\]

It can be easily checked that

\[
(\tau^{(n)}, \mu^{(n)})^T = T(\hat{\partial}, n) U
\]
Denote by $T_0(\hat{c}, n)$ the principal homogeneous part ($6 \times 6$ matrix) of the differential operator $T(\hat{c}, n)$, i.e.

$$T_0(\hat{c}, n) = \begin{bmatrix} T_0^{(1)}(\hat{c}, n) & T_0^{(2)}(\hat{c}, n) \\ T_0^{(3)}(\hat{c}, n) & T_0^{(4)}(\hat{c}, n) \end{bmatrix}_{6 \times 6}, \quad T_0^{(j)} = [T_{0pq}]_{3 \times 3}, \quad j = 1, 4$$

$$T_0^{(j)}(\hat{c}, n) = T_{pq}^{(j)}(\hat{c}, n) \quad \text{for} \quad j = 1, 3$$

$$T_0^{(2)}(\hat{c}, n) = T_{pq}^{(2)}(\hat{c}, n) + 2\alpha \sum_{k=1}^{3} \varepsilon_{pqk} n_k$$

$$T_0^{(4)}(\hat{c}, n) = T_{pq}^{(4)}(\hat{c}, n) + 2\nu \sum_{k=1}^{3} \varepsilon_{pqk} n_k$$

We have the evident equality

$$T(\hat{c}, n)U = T_0(\hat{c}, n)U + 2[zn \times \omega, vn \times \omega]^\top$$  \hspace{1cm} (2)

where the symbol $\times$ denotes the cross product of two vectors.

2.2. The basic equations

The basic equations of the hemitropic theory of elasticity in terms of the displacement and micro-rotation vectors read as follows [18]:

$$(\mu + \varepsilon)\Delta u(x) + (\lambda + \mu - \varepsilon) \text{grad} \, u(x) + (\kappa + \nu)\Delta \omega(x)$$

$$+ (\delta + \kappa - \nu) \text{grad} \, \omega(x) + 2\alpha \text{curl} \, \omega(x) + g\sigma^2 u(x) = -qF(x)$$

$$(\kappa + \nu)\Delta u(x) + (\delta + \kappa - \nu) \text{grad} \, u(x) + 2\alpha \text{curl} \, u(x) + (\gamma + \varepsilon)\Delta \omega(x)$$

$$+ (\beta + \gamma - \varepsilon) \text{grad} \, \omega(x) + 4\nu \text{curl} \, \omega(x) + (\mathcal{J}\sigma^2 - 4\alpha) \omega(x) = -gG(x)$$  \hspace{1cm} (3)

where $F = (F_1, F_2, F_3)^\top$ and $G = (G_1, G_2, G_3)^\top$ are the body force and body couple vectors per unit mass, $g > 0$ is the mass density of the elastic material, $\mathcal{J} > 0$ is a constant characterizing the so-called spin torque corresponding to the interior micro-rotations (i.e. the moment of inertia per unit volume), and $\sigma$ is a frequency parameter.

If $\sigma = \sigma_1 + i\sigma_2$ is a complex parameter with $\sigma_2 \neq 0$, then the above equations are called the pseudooscillation equations, while for $\sigma = 0$ they represent the equilibrium equations of statics.

Let us introduce the matrix differential operator corresponding to system (3):

$$L(\hat{c}, \sigma) := \begin{bmatrix} L^{(1)}(\hat{c}, \sigma) & L^{(2)}(\hat{c}, \sigma) \\ L^{(3)}(\hat{c}, \sigma) & L^{(4)}(\hat{c}, \sigma) \end{bmatrix}_{6 \times 6}$$  \hspace{1cm} (4)

where

$$L^{(1)}(\hat{c}, \sigma) := [(\mu + \varepsilon)\Delta + g\sigma^2]I_3 + (\lambda + \mu - \varepsilon)Q(\hat{c})$$

$$L^{(2)}(\hat{c}, \sigma) = L^{(3)}(\hat{c}, \sigma) := (\kappa + \nu)\Delta I_3 + (\delta + \kappa - \nu)Q(\hat{c}) + 2\alpha R(\hat{c})$$

$$L^{(4)}(\hat{c}, \sigma) := [(\gamma + \varepsilon)\Delta + (\mathcal{J}\sigma^2 - 4\alpha)]I_3 + (\beta + \gamma - \varepsilon)Q(\hat{c}) + 4\nu R(\hat{c})$$

Here $I_k$ stands for the $k \times k$ unit matrix and
\[
R(\partial) := \begin{bmatrix}
    0 & -\partial_3 & \partial_2 \\
    \partial_3 & 0 & -\partial_1 \\
    -\partial_2 & \partial_1 & 0
\end{bmatrix}_{3 \times 3}, \quad Q(\partial) := [\partial \partial_j]_{3 \times 3}
\]

It is easy to see that
\[
R(\partial)u = \text{curl} \, u, \quad Q(\partial)u = \text{grad} \, \text{div} \, u
\]

Due to the above notation, Equations (3) can be rewritten in matrix form as
\[
L(\partial, \sigma)U(x) = \Phi(x) \\
U = (u, \omega)^\top, \quad \Phi = (\Phi^{(1)}, \Phi^{(2)})^\top := (-\varrho F(x), -\varrho G(x))^\top
\]

Further, let us remark that the differential operator
\[
L(\partial) := L(\partial, 0)
\]

corresponds to the static equilibrium case, while the differential operator
\[
L_0(\partial) := \begin{bmatrix}
    L_0^{(1)}(\partial) & L_0^{(2)}(\partial) \\
    L_0^{(3)}(\partial) & L_0^{(4)}(\partial)
\end{bmatrix}_{6 \times 6}
\]

with
\[
L_0^{(1)}(\partial) := (\mu + \sigma)\Delta I_3 + (\lambda + \mu - \sigma) Q(\partial) \\
L_0^{(2)}(\partial) = L_0^{(3)}(\partial) := (\kappa + \nu)\Delta I_3 + (\delta + \kappa - \nu) Q(\partial) \\
L_0^{(4)}(\partial) := (\gamma + \varepsilon)\Delta I_3 + (\beta + \gamma - \varepsilon) Q(\partial)
\]

represents the principal homogeneous part of operators (4) and (5).

It is evident that
\[
L(\partial, \sigma)U - L(\partial)U = (\varrho \sigma^2 u, \varrho \sigma^2 \omega)^\top
\]

Let us remark that the operators $L(\partial, \sigma)$ for real $\sigma^2$, $L(\partial)$, and $L_0(\partial)$ are formally self-adjoint.

### 2.3. Green's formulae

For real-valued vectors $U := (u, \omega)^\top$, $U' := (u', \omega')^\top \in [C^2(\overline{\Omega^+})]^6$ there holds Green’s formula [18]
\[
\int_{\Omega^+} \left[ U' \cdot L(\partial)U + E(U', U) \right] \, dx = \int_{\partial \Omega^+} [U']^+ \cdot [T(\partial, n)U]^+ \, dS
\]
where $\Omega^+$ is a Lipschitz domain, $n$ is the outward unit normal vector to $\partial \Omega^+$, the symbols $\left[\cdot\right]^{\pm}$ denote the limits on $\partial$ from $\Omega^{\pm}$, $E(\cdot,\cdot)$ is the so-called energy bilinear form

$$
E(U',U) = E(U,U') = \sum_{p,q=1}^{3} \left\{ (\mu + \alpha)u_p'u_{pq} + (\mu - \alpha)u'_{pq}u_{qp} + (\kappa + \nu)(u'_{pq}\omega_{pq} + \omega'_{pq}u_{pq}) + (\kappa - \nu)(u'_{pq}\omega_{qp} + \omega'_{pq}u_{qp}) + (\gamma + \varepsilon)(\omega'_{pq}u_{pq} + \omega_{pq}u'_{pq})
\right.
$$

$$
+ (\gamma - \varepsilon)\omega'_{pq}\omega_{qp} + \delta(u'_{pq}\omega_{pq} + \omega'_{pq}u_{pq}) + \lambda u'_{pq}u_{qp} + \beta u'_{pq}u_{pq}\right\}
$$

(9)

with

$$
u_{pq} = \partial u_q - \sum_{k=1}^{3} \varepsilon_{pqk}\omega_k, \quad \omega_{pq} = \partial u_q, \quad p, q = 1, 2, 3
$$

(10)

Here and in what follows $a \cdot b$ denotes the usual scalar product of two (in general) complex vectors $a, b \in \mathbb{C}^m$:

$$a \cdot b = \sum_{j=1}^{m} a_j \bar{b}_j$$

where the over-bar denotes complex conjugation. The above Green formula immediately follows from the identity

$$U' \cdot L(\partial)U + E(U',U) = \sum_{p,q=1}^{3} \partial [u'_q \tau_{pq}(U) + \omega'_q \mu_{pq}(U)]$$

(11)

From (9) and (10) we get

$$E(U,U') = \frac{3\lambda + 2\mu}{3} \left( \text{div} \ u + \frac{3\delta + 2\kappa}{3\lambda + 2\mu} \text{div} \ \omega \right) \left( \text{div} \ u' + \frac{3\delta + 2\kappa}{3\lambda + 2\mu} \text{div} \ \omega' \right)$$

$$+ \frac{1}{3} \left( 3\beta + 2\gamma - \frac{(3\delta + 2\kappa)^2}{3\lambda + 2\mu} \right) \left( \text{div} \ \omega \right) \left( \text{div} \ \omega' \right)$$

$$+ \frac{\mu}{2} \sum_{k,j=1, k \neq j}^{3} \left[ \frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} + \frac{\kappa}{\mu} \left( \frac{\partial \omega_k}{\partial x_j} + \frac{\partial \omega_j}{\partial x_k} \right) \right]$$

$$\times \left[ \frac{\partial u'_k}{\partial x_j} + \frac{\partial u'_j}{\partial x_k} + \frac{\kappa}{\mu} \left( \frac{\partial \omega'_k}{\partial x_j} + \frac{\partial \omega'_j}{\partial x_k} \right) \right]$$

$$+ \frac{\mu}{3} \sum_{k,j=1}^{3} \left[ \frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k} + \frac{\kappa}{\mu} \left( \frac{\partial \omega_k}{\partial x_j} - \frac{\partial \omega_j}{\partial x_k} \right) \right]$$

$$\times \left[ \frac{\partial u'_k}{\partial x_j} - \frac{\partial u'_j}{\partial x_k} + \frac{\kappa}{\mu} \left( \frac{\partial \omega'_k}{\partial x_j} - \frac{\partial \omega'_j}{\partial x_k} \right) \right]$$

$$+ \left( \gamma - \frac{\kappa^2}{\mu} \right) \sum_{k,j=1, k \neq j}^{3} \left[ \frac{1}{2} \left( \frac{\partial \omega_k}{\partial x_j} + \frac{\partial \omega_j}{\partial x_k} \right) \left( \frac{\partial \omega'_k}{\partial x_j} + \frac{\partial \omega'_j}{\partial x_k} \right) \right].$$

\[ + \frac{1}{3} \left( \frac{\partial \omega_k}{\partial x_k} - \frac{\partial \omega_j}{\partial x_j} \right) \left( \frac{\partial \omega'_k}{\partial x_k} - \frac{\partial \omega'_j}{\partial x_j} \right) \]

\[ + \alpha \left( \nabla \mathbf{u} + \frac{v}{\alpha} \nabla \omega - 2 \mathbf{\omega} \right) \cdot \left( \nabla \mathbf{u}' + \frac{v}{\alpha} \nabla \omega' - 2 \mathbf{\omega}' \right) \]

\[ + \left( \varepsilon - \frac{\nu^2}{\alpha} \right) \nabla \omega \cdot \nabla \omega' \]

In particular,

\[ E(U, U) = \frac{3\lambda + 2\mu}{3} \left( \text{div} \mathbf{u} + \frac{3\delta + 2\kappa}{3\lambda + 2\mu} \text{div} \mathbf{\omega} \right)^2 \]

\[ + \frac{1}{3} \left( 3\beta + 2\gamma - \frac{(3\delta + 2\kappa)^2}{3\lambda + 2\mu} \right) \left( \text{div} \mathbf{\omega} \right)^2 \]

\[ + \frac{\mu}{2} \sum_{k,j=1,k\neq j} \left[ \frac{\partial u_k}{\partial x_j} - \frac{\partial u_j}{\partial x_k} + \frac{\kappa}{\mu} \left( \frac{\partial \omega_k}{\partial x_j} + \frac{\partial \omega_j}{\partial x_k} \right) \right]^2 \]

\[ + \left( \frac{\gamma - \kappa^2}{\mu} \right) \sum_{k,j=1,k\neq j} \left[ \frac{1}{2} \left( \frac{\partial \omega_k}{\partial x_j} + \frac{\partial \omega_j}{\partial x_k} \right)^2 + \frac{1}{3} \left( \frac{\partial \omega_k}{\partial x_k} - \frac{\partial \omega_j}{\partial x_j} \right)^2 \right] \]

\[ + \left( \varepsilon - \frac{\nu^2}{\alpha} \right) (\nabla \omega)^2 + \alpha \left( \nabla \mathbf{u} + \frac{v}{\alpha} \nabla \omega - 2 \mathbf{\omega} \right)^2 \] \hspace{1cm} (12)

From physical considerations (positive definiteness of the potential energy (12) with respect to variables (10)), it follows that the material constants satisfy the inequalities (cf. Reference [7])

\[ \mu > 0, \quad \alpha > 0, \quad 3\lambda + 2\mu > 0, \quad \mu \gamma - \kappa^2 > 0, \quad \alpha \varepsilon - \nu^2 > 0 \]

\[ (3\lambda + 2\mu)(3\beta + 2\gamma) - (3\delta + 2\kappa)^2 > 0 \] \hspace{1cm} (13)

whence

\[ \gamma > 0, \quad \varepsilon > 0, \quad \lambda + \mu > 0, \quad \beta + \gamma > 0, \quad d_1 := (\mu + \alpha)(\gamma + \varepsilon) - (\kappa + \nu)^2 > 0 \]

\[ d_2 := (\lambda + 2\mu)(\beta + 2\gamma) - (\delta + 2\kappa)^2 > 0 \]

Lemma 2.1

Let \( U = (u, \omega)^\top \in [C^1(\Omega)]^6 \) be a real-valued vector and \( E(U, U) = 0 \) in \( \Omega \). Then

\[ u(x) = [a \times x] + b, \quad \omega(x) = a, \quad x \in \Omega \]

where \( a \) and \( b \) are arbitrary three-dimensional constant vectors.
Proof
It easily follows from (12) and (13).

Throughout the paper $L_p$, $W^r_p$, $H^s_p$, and $B^r_{p,q}$ (with $r \geq 0$, $s \in \mathbb{R}$, $1 < p < \infty$, $1 \leq q \leq \infty$) denote the well-known Lebesgue, Sobolev–Slobodetski, Bessel potential, and Besov function spaces, respectively (see, e.g. References [29–31]). We will use the abbreviations $W^0_p = W_p$, $H^0 = H^1 = H^0 = L^2$. We recall that $H^s_p = W^s_p = B^s_{p,2}$, $H^s_p = B^s_{p,2}$, $W^r_p = B^r_{p,p}$, and $H^k_p = W^k_p$, for any $r \geq 0$, $s \in \mathbb{R}$, for any positive and non-integer $t$, and for any non-negative integer $k$.

If $U = U^{(1)} + iU^{(2)}$ is a complex-valued vector, where $U^{(j)} = (u^{(j)}, \omega^{(j)})^\top$ ($j = 1, 2$) are real-valued vectors, then

$$E(U, \bar{U}) = E(U^{(1)}, U^{(1)}) + E(U^{(2)}, U^{(2)})$$

and due to the positive definiteness of the energy form for real-valued vector functions with respect to variables (10), we have

$$E(U, \bar{U}) \geq c_0 \sum_{p,q=1}^{3} [(u^{(1)}_{pq})^2 + (u^{(2)}_{pq})^2 + (\omega p q^{(1)})^2 + (\omega p q^{(2)})^2]$$

$$\geq c_1 \sum_{j=1}^{2} \sum_{p,q=1}^{3} [(\partial_p u^{(j)})^2 + (\partial_p \omega^{(j)})^2] - c_2 \sum_{j=1}^{2} \sum_{p=1}^{3} [\omega p^{(j)}]^2$$

(14)

where $c_0$, $c_1$, and $c_2$ are positive constants depending only on the material constants, and $u^{(j)}$ and $\omega^{(j)}$ are defined by formulae (10) with $u^{(j)}$ and $\omega^{(j)}$ for $u$ and $\omega$.

From (14) it follows that for an arbitrary real-valued vector $U \in [H^1(\Omega^+)]^6$ and a Lipschitz domain $\Omega^+$

$$\mathcal{B}_{u^+}(U, U) := \int_{\Omega^+} E(U, \bar{U}) \, dx \geq c_1 \int_{\Omega^+} \left\{ \sum_{p,q=1}^{3} [(\partial_p u_q)^2 + (\partial_p \omega_q)^2] \right\} \, dx$$

$$- c_2 \int_{\Omega^+} \sum_{p=1}^{3} [u^2_p + \omega^2_p] \, dx$$

i.e. the following Korn’s type inequality holds (cf. References [32, Part I, Section 12; 25, Chapter 10])

$$\mathcal{B}_{u^+}(U, U) \geq c_1^* \|U\|_{[H^1(\Omega^+)]^6}^2 - c_2^* \|U\|_{[H^1(\Omega^+)]^6}^2$$

(15)

where $\|\cdot\|_{[H^s(\Omega)]^6}$ denotes the norm in the Sobolev space $[H^s(\Omega)]^6$, and $c_1^*$ and $c_2^*$ are positive constants depending only on the material constants.

These results imply that the differential operators $L(\partial, \sigma)$, $L(\partial)$, and $L(\partial)$ are strongly elliptic and the following inequality (the accretivity condition) holds (cf. e.g. References [32, Part I, Section 5; 25, Chapter 4, Lemma 4.5])

$$c_1^* |\xi|^2 |\eta|^2 \geq L_0(\xi) \eta \cdot \eta = \sum_{k,j=1}^{6} \{L_0(\xi_k)\}_{ij} \eta_i \eta_j \geq c_1^* |\xi|^2 |\eta|^2$$

with some constants $c_k^* > 0$ ($k = 1, 2$) for arbitrary $\xi \in \mathbb{R}^3$ and arbitrary complex vector $\eta \in \mathbb{C}^6$. 

Remark 2.2
For a Lipschitz domain \( \Omega^+ \), an arbitrary complex parameter \( \sigma \) and \( U, U' \in [C^2(\overline{\Omega^+})]^6 \), we have (cf. (7)–(8))
\[
\int_{\Omega^+} [U' \cdot L(\partial, \sigma)\overline{U} - L(\partial, \sigma)U' \cdot \overline{U}] \, dx
\]
\[
= \int_{\partial \Omega^+} \{[U']^+ \cdot [T(\partial, n)\overline{U}^+] - [T(\partial, n)U']^+ \cdot [\overline{U}^+] \} \, dS
\]

Remark 2.3
By standard arguments, Green’s formula (8) can be extended to the case of complex-valued vector functions \( U \in [W^1_p(\Omega^+)]^6 \) and \( U' \in [W^1_p(\Omega^+)]^6 \) with \( 1/p + 1/p' = 1 \) and \( L(\partial)U \in [L^p(\Omega^+)]^6 \) (cf. References [25,31,33,34])
\[
\int_{\Omega^+} [U' \cdot L(\partial)U + E(U', \overline{U})] \, dx = \langle [U']^+, [\overline{T(\partial, n)U}^+] \rangle_{\partial \Omega^+}
\]
where \( [U']^+ \in [B^{1-1/p'}_{p', p'}(\partial \Omega^+)]^6 \) is the trace of \( U' \) on \( \partial \Omega^+ \) and \( \langle \cdot, \cdot \rangle_{\partial \Omega^+} \) denotes the duality between the spaces \([B^{1-1/p'}_{p', p'}(\partial \Omega^+)]^6 \) and \([B^{1-1/p'}_{p', p'}(\partial \Omega^+)]^6 \), which extends the usual \( L^2 \)-scalar product for regular vector-functions, i.e. for \( f, g \in [L^2(S)]^6 \) we have
\[
\langle f, g \rangle_S = \sum_{k=1}^6 \int_S f_k g_k \, dS = : (f, g)_{[L^2(S)]^6}
\]
Clearly, in this case the functional \( [T(\partial, n)U]^+ \in [B^{1/p'}_{p', p'}(\partial \Omega^+)]^6 \) is correctly determined by relation (16). This functional will be referred to as a trace of the stress vector on \( \partial \Omega^+ \).

Remark 2.4
For the operator \( L_0(\partial) \) we have the following Green’s formula for arbitrary real-valued vector functions \( U := (u, \omega)^\top, U' := (u', \omega')^\top \in [H^1(\Omega^+)]^6 \) and \( L_0(\partial)U \in [H^0(\Omega^+)]^6 \)
\[
\int_{\Omega^+} [L_0(\partial)U \cdot U' + E_0(U, U')] \, dx = \int_{\partial \Omega^+} [T_0(\partial, n)U]^+ \cdot [U']^+ \, dS
\]
where \( n \) is the outward unit normal vector to \( \partial \Omega^+ \),
\[
E_0(U, U') = E_0(U', U) = \frac{3\lambda + 2\mu}{3} \left( \text{div} \, u + \frac{3\delta + 2\kappa}{3\lambda + 2\mu} \text{div} \, \omega \right) \left( \text{div} \, u' + \frac{3\delta + 2\kappa}{3\lambda + 2\mu} \text{div} \, \omega' \right) + \frac{1}{3} \left( 3\beta + 2\gamma - \frac{(3\delta + 2\kappa)^2}{3\lambda + 2\mu} \right) \left( \text{div} \, \omega \right) \left( \text{div} \, \omega' \right)
\]
\[
+ \frac{\mu}{2} \sum_{k,j=1, k \neq j}^3 \left[ \frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} + \frac{\kappa}{\mu} \left( \frac{\partial \omega_k}{\partial x_j} + \frac{\partial \omega_j}{\partial x_k} \right) \right] \times \left[ \frac{\partial u'_k}{\partial x_j} + \frac{\partial u'_j}{\partial x_k} + \frac{\kappa}{\mu} \left( \frac{\partial \omega'_k}{\partial x_j} + \frac{\partial \omega'_j}{\partial x_k} \right) \right]
\]
\[ + \frac{\mu}{3} \sum_{k,j=1}^{3} \left[ \frac{\partial u_k}{\partial x_k} - \frac{\partial u_j}{\partial x_j} + \kappa \left( \frac{\partial \omega_k}{\partial x_k} - \frac{\partial \omega_j}{\partial x_j} \right) \right] \]
\[
\times \left[ \frac{\partial u'_k}{\partial x_k} - \frac{\partial u'_j}{\partial x_j} + \kappa \left( \frac{\partial \omega'_k}{\partial x_k} - \frac{\partial \omega'_j}{\partial x_j} \right) \right] \]
\[+ \left( \gamma - \frac{\kappa^2}{\mu} \right) \sum_{k,j=1, k \neq j}^{3} \left[ \frac{1}{2} \left( \frac{\partial \omega_k}{\partial x_j} + \frac{\partial \omega_j}{\partial x_k} \right) \left( \frac{\partial \omega'_k}{\partial x_j} + \frac{\partial \omega'_j}{\partial x_k} \right) \right] \]
\[+ \frac{1}{3} \left( \frac{\partial \omega_k}{\partial x_k} - \frac{\partial \omega_j}{\partial x_j} \right) \left( \frac{\partial \omega'_k}{\partial x_k} - \frac{\partial \omega'_j}{\partial x_j} \right) \]
\[\times \left( \text{curl } u + \frac{v}{\alpha} \text{ curl } \omega \right) \cdot \left( \text{curl } u' + \frac{v}{\alpha} \text{ curl } \omega' \right) \]
\[+ \left( \frac{\varepsilon - \frac{v^2}{\alpha}}{\alpha} \right) \text{ curl } \omega \cdot \text{ curl } \omega' \]

In particular,

\[
E_0(U, U) = \frac{2\lambda + 2\mu}{3} \left( \text{div } u + \frac{3\delta + 2\kappa}{2\lambda + 2\mu} \text{ div } \omega \right)^2 \]
\[+ \frac{1}{3} \left( 3\beta + 2\gamma - \frac{(3\delta + 2\kappa)^2}{2\lambda + 2\mu} \right) (\text{div } \omega)^2 \]
\[+ \frac{\mu}{2} \sum_{k,j=1, k \neq j}^{3} \left[ \frac{\partial u_k}{\partial x_j} + \frac{\partial u_j}{\partial x_k} + \kappa \left( \frac{\partial \omega_k}{\partial x_j} + \frac{\partial \omega_j}{\partial x_k} \right) \right]^2 \]
\[+ \left( \gamma - \frac{\kappa^2}{\mu} \right) \sum_{k,j=1, k \neq j}^{3} \left[ \frac{1}{2} \left( \frac{\partial \omega_k}{\partial x_j} + \frac{\partial \omega_j}{\partial x_k} \right) \right] \left( \frac{\partial \omega'_k}{\partial x_k} - \frac{\partial \omega'_j}{\partial x_j} \right) \]
\[+ \left( \frac{\varepsilon - \frac{v^2}{\alpha}}{\alpha} \right) (\text{curl } \omega)^2 + \alpha \left( \text{curl } u + \frac{v}{\alpha} \text{ curl } \omega \right)^2 \]

(17)

Whence we easily derive that the equality \( E_0(U, U) = 0 \) in \( \Omega \subset \mathbb{R}^3 \) implies

\[ U(x) = (b', b'')^\top \text{ in } \Omega \]

where \( b' \) and \( b'' \) are arbitrary three-dimensional constant vectors.
If \( U, U' \in H^1_{\text{loc}}(\Omega^-) \), \( L_0(\partial)U \in H^0_{\text{loc}}(\Omega^-) \), \( \partial^2 U, \partial^2 U' = \mathcal{O}(|x|^{-1-|\alpha|}) \) as \( |x| \to +\infty \) for multi-indices \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) with \( |\alpha| = 0, 1 \), then there holds Green's formula

\[
\int_{\Omega^-} [L_0(\partial)U \cdot U' + E_0(U, \overline{U'})] \, dx = -\int_{\partial \Omega^-} [T_0(\partial, n)U]^- \cdot [U']^- \, dS
\]

**Remark 2.5**

Note that if \( U, U' \in [H^1_{\text{loc}}(\Omega^-)]^6 \), \( L(\partial, \sigma)U \in [H^0(\Omega^-)]^6 \cap [L_1(\Omega^-)]^6 \)

\[
\partial^2 u, \partial^2 u' = \mathcal{O}(|x|^{-1-|\alpha|}), \quad \partial^2 \omega, \partial^2 \omega' = \mathcal{O}(|x|^{-2})
\]

as \( |x| \to +\infty \) for multi-indices \( \alpha = (\alpha_1, \alpha_2, \alpha_3) \) with \( |\alpha| = 0, 1 \), then there holds Green's formula for an exterior domain \( \Omega^- \)

\[
\int_{\Omega^-} [L(\partial)U \cdot U' + E(U, \overline{U'})] \, dx = \int_{\Omega^-} [L(\partial, \sigma)U \cdot U' + E^{(\sigma)}(U, \overline{U'})] \, dx
\]

\[
= -\int_{\partial \Omega^-} [T(\partial, n)U]^- \cdot [U']^- \, dS
\]

where

\[
E^{(\sigma)}(U, \overline{U'}) = E(U, \overline{U'}) - \rho \sigma^2 |u| - \mathcal{I} \sigma^2 \omega \cdot \overline{\omega'}
\]

Here either \( \sigma^2 \) is a sufficiently large negative number or \( \sigma = 0 \).

**Remark 2.6**

Let

\[
E^{(\sigma)}(U, U) := E(U, U) - \rho \sigma^2 |u|^2 - \mathcal{I} \sigma^2 |\omega|^2
\]

and

\[
\mathcal{B}^{(\sigma)}_\Omega(U, U) := \int_{\Omega} E^{(\sigma)}(U, U) \, dx, \quad \Omega \in \{\Omega^+, \Omega^-, \Omega^+_\Omega^-\}
\]

In view of (15) it is evident that for a sufficiently large negative number \( \sigma^2 < 0 \) \( (-\sigma^2 > \mathcal{B}^{-1} + \mathcal{I}^{-1}) \) \( c^* \) say) we have

\[
\mathcal{B}^{(\sigma)}_\Omega(U, U) \geq c^* \|U\|_{H^1(\Omega)}^2
\]

for all \( U \in [H^1(\Omega)]^6 \), where \( \Omega \in \{\Omega^+, \Omega^-, \Omega^+_\Omega^-\} \).

### 2.4. Weak formulation of boundary value problems

We assume that \( S = \partial \Omega^\pm \) is a Lipschitz boundary and either \( \sigma = 0 \) or \( \sigma^2 \) is a sufficiently large negative number.
The weak formulation of the Dirichlet \((I^{(o)})^\pm\) and Neumann \((II^{(o)})^\pm\) boundary value problems reads as follows:

Find a distributional solution \(U \in [H^1(\Omega^\pm)]^6\) or \(U \in [H^1_{\text{loc}}(\Omega^\pm)]^6\) of the differential equation

\[
L(\partial, \sigma)U = 0 \quad \text{in} \ \Omega^\pm
\]

satisfying either the Dirichlet condition

\[
[U]^\pm = f \quad \text{on} \ S \tag{18}
\]

or the Neumann condition

\[
[T(\partial, n)U]^\pm = F \quad \text{on} \ S \tag{19}
\]

where \(f \in [H^{1/2}(S)]^6\) and \(F \in [H^{-1/2}(S)]^6\). Moreover in the case of unbounded domain \(\Omega^-\) we assume that the decay conditions at infinity mentioned in Remark 2.5 are satisfied.

The boundary condition (18) is understood in the usual trace sense, while (19) is understood in the functional sense described in Remark 2.3.

There holds the following uniqueness result (see Reference [18]).

**Theorem 2.7**

The homogeneous versions of the Dirichlet and Neumann boundary value problems \((I^{(o)})^\pm_0\), \((I^{(o)})^\pm\), \((II^{(o)})^\pm_0\), and \((II^{(o)})^\pm\) have only the trivial solution provided \(\sigma \neq 0\). A general solution of the BVP \((II^{(o)})^+_0\) is a vector \(U = (a \times x + b, a)^\top\), where \(a\) and \(b\) are as in Lemma 2.1.

### 3. BASIC FUNDAMENTAL MATRICES

Denote by \(\Gamma(x, \sigma) = [\Gamma_{kj}(x, \sigma)]_{6 \times 6}\) the matrix of fundamental solutions of the differential operator \(L(\partial, \sigma)\)

\[
L(\partial, \sigma) \Gamma(x, \sigma) = \delta(x)I_6
\]

This matrix is constructed explicitly in Reference [18], where it is shown that

(a) the entries of the matrix \(\Gamma(x, \sigma)\) and its derivatives are \(C^\infty\)-regular in \(\mathbb{R}^3 \setminus \{0\}\) and decay exponentially as \(|x| \to +\infty\) if \(\sigma^2\) is a sufficiently large negative number;

(b) if \(\sigma = 0\) then the derivatives of order \(m (m \geq 0)\) of the entries of \(\Gamma(x, 0)\) decay as \(\mathcal{C}(|x|^{-m-1})\) as \(|x| \to +\infty\). Moreover, \(\partial^2 \Gamma_{kj}(x, 0) = \mathcal{C}(|x|^{-2-|k|})\) as \(|x| \to +\infty\) for either \(k = 4, 5, 6\) or \(j = 4, 5, 6\).

(c) the fundamental matrix has the property

\[
\Gamma(x, \sigma) = \frac{1}{8\pi^3} \int_{\mathbb{R}^3} [L(-i\xi, \sigma)]^{-1} e^{-ix\xi} d\xi = [\Gamma(-x, \sigma)]^\top
\]

where \(L(-i\xi, \sigma)\) is the symbol matrix of the operator \(L(\partial, \sigma)\) and the above formal integral is understood as a generalized Fourier transform.

The principal singular part of the fundamental matrix of the operator \(L(\partial, \sigma)\) we denote by \(\Gamma_0(x)\). It represents a fundamental matrix of the operator \(L_0(\partial)\) defined by (6) and solves

\[
L_0(\partial) \Gamma_0(x) = \delta(x)I_6
\]
The explicit form of the matrix is [18]

\[
\Gamma_0(x) = -\frac{1}{8\pi d_1 d_2 |x|^2} \left\{ \begin{bmatrix} a_1 I_3 & a_2 I_3 \\ a_2 I_3 & a_3 I_3 \end{bmatrix} - \frac{1}{|x|^2} \begin{bmatrix} b_1 Q(x) & b_2 Q(x) \\ b_2 Q(x) & b_3 Q(x) \end{bmatrix} \right\}
\]

\[
a_1 := d_2(\gamma + \varepsilon) + d_1(\beta + 2\gamma), \quad b_1 := d_1(\beta + 2\gamma) - d_2(\gamma + \varepsilon)
\]

\[
a_2 := -[d_2(\kappa + \nu) + d_1(\delta + 2\kappa)], \quad b_2 := -[d_1(\delta + 2\kappa) - d_2(\kappa + \nu)]
\]

\[
a_3 := d_2(\mu + \alpha) + d_1(\lambda + 2\mu), \quad b_3 := d_1(\lambda + 2\mu) - d_2(\mu + \alpha)
\]

We can easily see that \(\Gamma_0(x)\) is symmetric, its entries are even, homogeneous functions of order \(-1\) and in a vicinity of the origin (i.e. for small \(|x|\))

\[
\partial^\alpha [\Gamma(x, \sigma) - \Gamma_0(x)] = O(|x|^{-|\alpha|})
\]

for an arbitrary multi-index \(\alpha = (\alpha_1, \alpha_2, \alpha_3)\) and an arbitrary complex number \(\sigma\).

4. GENERAL INTEGRAL REPRESENTATIONS

In what follows we assume that \(S = \partial \Omega^\pm\) is a Lipschitz boundary, i.e. \(S \subset C^{0,1}\), and \(n(x)\) stands for the outward unit normal vector to \(\Omega^+\) at the point \(x \in S\). Clearly, \(n\) is defined almost everywhere on \(S\) and \(n \in L_\infty(S)\).

Let us introduce the generalized single- and double-layer potentials, and the Newtonian volume potential

\[
V^{(\sigma)}(\varphi)(x) = \int_S \Gamma(x - y, \sigma) \varphi(y) \, dS_y, \quad x \in \mathbb{R}^3 \setminus S \tag{21}
\]

\[
W^{(\sigma)}(\varphi)(x) = \int_S [T(\partial_y n(y)) \Gamma(y - x, \sigma)]^\top \varphi(y) \, dS_y, \quad x \in \mathbb{R}^3 \setminus S \tag{22}
\]

\[
N_\Omega^{(\sigma)}(\psi)(x) = \int_\Omega \Gamma(x - y, \sigma) \psi(y) \, dy, \quad x \in \mathbb{R}^3
\]

where \(T(\partial, n)\) is the stress operator of the theory of hemitropic elasticity (see (1)), \(\Gamma(\cdot, \sigma)\) is the fundamental matrix, \(\varphi = (\varphi_1, \ldots, \varphi_6)^\top\) is a density vector-function defined on \(S\), while a density vector-function \(\psi = (\psi_1, \ldots, \psi_6)^\top\) is defined on \(\Omega \in \{\Omega^+, \Omega^-\}\).

It can easily be checked that the potentials defined by (21) and (22) are \(C^\infty\)-smooth in the domain \(\mathbb{R}^3 \setminus S\) and solve the homogeneous equations (3) \((F = 0, G = 0)\) for an arbitrary \(L_p\)-summable vector function \(\varphi\). The volume potential \(N_\Omega^{(\sigma)}(\psi)\) solves the non-homogeneous equation \(L(\partial, \sigma) U(x) = \psi(x)\) almost everywhere in \(\Omega^+\) for \(\psi \in [L_2(\Omega^+)]^6\) and in \(\Omega^-\) for \(\psi \in [L_2, \text{comp}(\Omega^-)]^6\).

By standard arguments we can prove the following assertions (cf. e.g. References [25,31,34, 35, Chapter I, Lemma 2.1; Chapter II, Lemma 8.2]).

**Theorem 4.1**

Let \(S\) be a Lipschitz boundary and \(U \in [H^1(\Omega^+)]^6\) with \(L(\partial, \sigma) U \in [L_2(\Omega^+)]^6\).
Then there holds the following integral representation formula

\[
W^{(\sigma)}([U]^+)(x) - V^{(\sigma)}([TU]^+)(x) + K^{(\sigma)}_{\Omega}(L(\partial, \sigma)U)(x) = \begin{cases} U(x) & \text{for } x \in \Omega^+ \\ 0 & \text{for } x \in \Omega^- \end{cases} \tag{23}
\]

**Remark 4.2**

Let \( S \) be a Lipschitz boundary and \( \sigma^2 \) be a sufficiently large negative number. Then it can be shown that every polynomially bounded solution to the equation \( L(\partial, \sigma)U = 0 \) in \( \Omega^- \) actually decays (together with its derivatives) at infinity exponentially (see Reference [18]):

\[
U(x) = \mathcal{O}(\exp\{-c_0 |x|\}) \quad \text{as} \quad |x| \to +\infty
\]

where \( c_0 = \text{const} > 0 \).

Therefore, if \( U \in [H^1(\Omega^-)]^6 \) solves the equation \( L(\partial, \sigma)U = 0 \) in \( \Omega^- \), then there holds the following integral representation formula:

\[
-W^{(\sigma)}([U]^-)(x) + V^{(\sigma)}([TU]^-)(x) = \begin{cases} U(x) & \text{for } x \in \Omega^- \\ 0 & \text{for } x \in \Omega^+ \end{cases}
\]

### 5. Properties of Potentials and Boundary Operators

The jump and mapping properties of the above-introduced single- and double-layer potentials and the corresponding boundary integral (pseudodifferential) operators in the Hölder \( (C^{k+\epsilon}) \), Sobolev–Slobodetski \( (W_p^s) \), Bessel potential \( (H_p^s) \) and Besov \( (B_{p,q}^s) \) function spaces can be studied by standard methods (see, e.g. References [18,25,35–44]).

We recall here some results from [18] concerning the properties of the single- and double-layer potentials.

**Theorem 5.1**

Let \( S \in C^{k+1,\gamma_0} \) where \( k \geq 0 \) is an integer, \( 0 < \gamma \leq 1 \), and let \( 0 < \gamma_0 < \gamma \). Then the operators

\[
V^{(\sigma)} : [C^{k,\gamma_0}(S)]^6 \to [C^{k+1,\gamma_0}(\Omega^\pm)]^6
\]

\[
W^{(\sigma)} : [C^{k,\gamma_0}(S)]^6 \to [C^{k,\gamma_0}(\Omega^\pm)]^6
\]

are bounded.

For any \( g \in C^{k,\gamma_0}(S) \) and any \( x \in S \)

\[
[V^{(\sigma)}(g)(x)]^\pm = V^{(\sigma)}(g)(x) = \mathcal{H}^{(\sigma)}g(x) \tag{24}
\]

\[
[T(\partial, n(x))V^{(\sigma)}(g)(x)]^\pm = [\pm 2^{-1}I_6 + \mathcal{H}^{(\sigma)}]g(x) \tag{25}
\]

\[
[W^{(\sigma)}(g)(x)]^\pm = [\pm 2^{-1}I_6 + \mathcal{H}^{(\sigma)}]g(x) \tag{26}
\]

\[
[T(\partial, n(x))W^{(\sigma)}(g)(x)]^\pm = [T(\partial, n(x))W^{(\sigma)}(g)(x)]^- = \mathcal{L}^{(\sigma)}g(x), \quad k \geq 1 \tag{27}
\]
where
\[
\mathcal{K}^{(\sigma)} g(x) := \int_S \Gamma(x - y, \sigma) g(y) \, dS_y
\]
\[
\mathcal{K}^{(\sigma)} g(x) := \int_S T(\partial_x, n(x)) \Gamma(x - y, \sigma) g(y) \, dS_y
\]
\[
\mathcal{K}^{(\sigma)\ast} g(x) := \int_S [T(\partial_y, n(y)) \Gamma(y - x, \sigma)]^\top g(y) \, dS_y
\]
\[
\mathcal{L}^{(\sigma)} g(x) := \lim_{\Omega \ni x \rightarrow S} T(\partial_z, n(x)) \int_S [T(\partial_y, n(y)) \Gamma(y - z, \sigma)]^\top g(y) \, dS_y
\]  
(28)

By standard arguments we can extend these results to the case of Lipschitz domains (cf. References [19,24,25,27,28,33,45]).

**Theorem 5.2**

Let \( S \) be a Lipschitz boundary. The operators \( V^{(\sigma)} \) and \( W^{(\sigma)} \) can be extended by continuity to the bounded mappings

\[
V^{(\sigma)} : [H^{-1/2}(S)]^6 \rightarrow [H^1(\Omega^+)]^6 \quad [[H^{-1/2}(S)]^6 \rightarrow [H^1_{\text{loc}}(\Omega^-)]^6]
\]
\[
W^{(\sigma)} : [H^{1/2}(S)]^6 \rightarrow [H^1(\Omega^+)]^6 \quad [[H^{1/2}(S)]^6 \rightarrow [H^1_{\text{loc}}(\Omega^-)]^6]
\]

The jump relations (24)–(27) on \( S \) remain valid for the extended operators in the corresponding function spaces.

Moreover, \( V^{(\sigma)}(g) \in [H^1(\mathbb{R}^3)]^6 \) for \( g \in [H^{-1/2}(S)]^6 \) and \( W^{(\sigma)}(h) \in [H^1(\Omega^-)]^6 \) for \( h \in [H^{1/2}(S)]^6 \) if \( \sigma^2 \) is a sufficiently large negative number.

**Theorem 5.3**

Let \( S \) be a Lipschitz boundary. Then the operators

\[
\mathcal{K}^{(\sigma)} : [H^{-1/2}(S)]^6 \rightarrow [H^{1/2}(S)]^6
\]
\[
\pm 2^{-1} I_6 + \mathcal{K}^{(\sigma)} : [H^{-1/2}(S)]^6 \rightarrow [H^{-1/2}(S)]^6
\]
\[
\pm 2^{-1} I_6 + \mathcal{K}^{(\sigma)\ast} : [H^{1/2}(S)]^6 \rightarrow [H^{1/2}(S)]^6
\]
\[
\mathcal{L}^{(\sigma)} : [H^{1/2}(S)]^6 \rightarrow [H^{-1/2}(S)]^6
\]  
(29)

are bounded.

Moreover, \( \mathcal{K}^{(\sigma)} \) and \( \mathcal{L}^{(\sigma)} \) are self-adjoint and the following equalities hold in appropriate function spaces (associated with the mappings (29)–(32)):

\[
\mathcal{K}^{(\sigma)\ast} \mathcal{K}^{(\sigma)} = \mathcal{K}^{(\sigma)} \mathcal{K}^{(\sigma)\ast}, \quad \mathcal{L}^{(\sigma)} \mathcal{K}^{(\sigma)\ast} = \mathcal{K}^{(\sigma)} \mathcal{L}^{(\sigma)}
\]
\[
\mathcal{K}^{(\sigma)} \mathcal{L}^{(\sigma)} = - 4^{-1} I_6 + (\mathcal{K}^{(\sigma)\ast})^2, \quad \mathcal{L}^{(\sigma)} \mathcal{K}^{(\sigma)} = - 4^{-1} I_6 + (\mathcal{K}^{(\sigma)})^2
\]  
(33)

**Theorem 5.4**

Let \( S \) be a Lipschitz boundary and \( \sigma^2 \) be a sufficiently large negative number.
Then the operators
\[
\mathcal{H}^{(\sigma)} : [H^{-1/2}(S)]^6 \to [H^{1/2}(S)]^6
\]
\[
\mathcal{L}^{(\sigma)} : [H^{1/2}(S)]^6 \to [H^{-1/2}(S)]^6
\]
are isomorphisms.

The vectors \( U_I = V^{(\sigma)}([\mathcal{H}^{(\sigma)}]^{-1} f) \in [H^1(\Omega^{\pm})]^6 \) and \( U_H = W^{(\sigma)}([\mathcal{L}^{(\sigma)}]^{-1} F) \in [H^1(\Omega^{\pm})]^6 \) represent unique solutions to the BVPs \((I^{(\sigma)})_f^+\) and \((II^{(\sigma)})_F^\pm\) provided \( f \in [H^{1/2}(S)]^6 \) and \( F \in [H^{-1/2}(S)]^6 \).

**Proof**
The first part of the theorem follows from Theorem 5.3 and the well-known Lax-Milgram lemma since
\[
\langle -\mathcal{H}^{(\sigma)} g, g \rangle_S \geq \delta_1 \|g\|_{[H^{-1/2}(S)]^6}^2 \quad \forall g \in [H^{-1/2}(S)]^6
\]
\[
\langle \mathcal{L}^{(\sigma)} h, h \rangle_S \geq \delta_2 \|h\|_{[H^{1/2}(S)]^6}^2 \quad \forall h \in [H^{1/2}(S)]^6
\]
which can be shown by the standard arguments (cf. References [25,33,34]).

The second part of the theorem is a consequence of Theorems 5.2 and 2.7.

**Corollary 5.5**
Let \( S = \partial \Omega^{\pm} \) be a Lipschitz boundary and \( \sigma^2 \) be a sufficiently large negative number. Further, let \( U \in [H^1(\Omega^{\pm})]^6 \) be a solution to the equation \( L(\hat{\nu}, \sigma) U = 0 \).

Then there exist positive constants \( C_1 \) and \( C_2 \), depending on the hemitropic constants and Lipschitz parameters of the domain \( \Omega^{\pm} \), such that
\[
\|U\|_{[H^1(\Omega^{\pm})]^6} \leq C_1 \|\xi^{(\pm)}_S[U]\|_{[H^{1/2}(S)]^6}
\]
\[
\|U\|_{[H^1(\Omega^{\pm})]^6} \leq C_2 \|\xi^{(\pm)}_S[TU]\|_{[H^{-1/2}(S)]^6}
\]
where \( \xi^{(\pm)}_S \) is the corresponding trace operator on \( S \):
\[
\xi^{(\pm)}_S[\cdot] := [\cdot]^{\pm}_S.
\]

**Proof**
The proof follows from Theorems 5.4 and 5.2.

### 6. Auxiliary Estimates

In this section, we will establish some regularity properties of solutions to the basic BVPs. To this end, first we derive several auxiliary formulas which are crucial to obtain improved smoothness results for mechanical characteristics of the hemitropic elastic field (cf. References [19,21,24,25,33]).

#### 6.1. Rellich–Payne–Weinberger identity

We start with the following:

**Lemma 6.1**
Assume that \( \Omega^+ \) is a Lipschitz domain. For an arbitrary vector-function \( h = (h_1, h_2, h_3) \in [C^1(\mathbb{R}^3)]^3 \) with a compact support and a vector-function \( U \in [H^2(\Omega^+)]^6 \) there holds
the identity
\[
\int_S \{(n \cdot h)E(U, U)\}^+ \, dS = 2 \int_S \{h_1 \hat{\partial}_1 U \cdot TU\}^+ \, dS + \int_{\Omega^+} (\text{div } h)E(U, U) \, dx
\]
\[
- 2 \int_{\Omega^+} h_1 \hat{\partial}_1 U \cdot \Phi \, dx - 2 \int_{\Omega^+} (\hat{\partial}_p h_1)(\hat{\partial}_1 u_q)\tau_{pq}(U)
\]
\[
+ (\hat{\partial}_1 \omega_q)\mu_{pq}(U) \big] \, dx
\]  \hfill (34)

where \( \Phi(x) := L(\partial)U(x) \in [L_2(\Omega^+)]^6 \).

**Proof**

We have the evident identity
\[
\int_{\Omega^+} \text{div } [hE(U, U)] \, dx = \int_S \{(n \cdot h)E(U, U)\}^+ \, dS
\]  \hfill (35)

Clearly,
\[
\text{div } [hE(U, U)] = E(U, U) \text{ div } h + h_1 \hat{\partial}_1 E(U, U)
\]  \hfill (36)

With the help of (14) it can be easily seen that
\[
\hat{\partial}_1 E(U, U) = 2E(\hat{\partial}_1 U, U) = 2\hat{\partial}_p[((\hat{\partial}_1 u_q)\tau_{pq}(U) + (\hat{\partial}_1 \omega_q)\mu_{pq}(U))] - 2\hat{\partial}_1 U \cdot L(\partial)U
\]  \hfill (37)

Further, we have
\[
h_1\hat{\partial}_1 E(U, U) = -2h_1\hat{\partial}_1 U \cdot L(\partial)U + 2h_1\hat{\partial}_p[((\hat{\partial}_1 u_q)\tau_{pq}(U) + (\hat{\partial}_1 \omega_q)\mu_{pq}(U))]
\]
\[
= -2h_1\hat{\partial}_1 U \cdot L(\partial)U + 2\hat{\partial}_p[h_1(\hat{\partial}_1 u_q)\tau_{pq}(U) + h_1(\hat{\partial}_1 \omega_q)\mu_{pq}(U)]
\]
\[
- 2(\hat{\partial}_p h_1)\tau_{pq}(U) + (\hat{\partial}_1 \omega_q)\mu_{pq}(U) \big] \]  \hfill (38)

Relations (35)–(38) imply
\[
\int_S \{(n \cdot h)E(U, U)\}^+ \, dS = \int_{\Omega^+} \text{div } [hE(U, U)] \, dx
\]
\[
= \int_{\Omega^+} (\text{div } h)E(U, U) \, dx + 2 \int_{\Omega^+} \hat{\partial}_p[h_1(\hat{\partial}_1 u_q)\tau_{pq}(U)]
\]
\[
+ h_1(\hat{\partial}_1 \omega_q)\mu_{pq}(U) \big] \, dx - 2 \int_{\Omega^+} (\hat{\partial}_p h_1)(\hat{\partial}_1 u_q)\tau_{pq}(U)
\]
\[
+ (\hat{\partial}_1 \omega_q)\mu_{pq}(U) \big] \, dx - 2 \int_{\Omega^+} h_1\hat{\partial}_1 U \cdot \Phi \, dx
\]
\[
= \int_{\Omega^+} (\text{div } h)E(U, U) \, dx + 2 \int_{\Omega^+} n_p[h_1(\hat{\partial}_1 u_q)\tau_{pq}(U)]
\]
\[ + h_{i}(\partial_{i} \omega_{q}) \mu_{pq}(U) + 2 \int_{\Omega^{+}} (\partial_{p} h_{i}) \left[ (\partial_{i} u_{q}) \tau_{pq}(U) \right] dS \]

\[ - (\partial_{i} \omega_{q}) \mu_{pq}(U) \int_{\Omega^{+}} h_{i} \partial_{i} U \cdot \Phi \, dx \]

which completes the proof.

**Corollary 6.2**
Assume that \( \Omega^{-} \) is a Lipschitz domain. For an arbitrary vector \( h = (h_{1}, h_{2}, h_{3}) \in [C^{1}(\mathbb{R}^{3})]^{3} \) with a compact support and a vector \( U \in [H^{2}_{\text{loc}}(\Omega^{-})]^{6} \) there holds the identity

\[ - \int_{S} \{(n \cdot h) E(U, U)\}^{-} dS = -2 \int_{S} \{h_{i} \partial_{i} U \cdot TU\}^{-} dS + \int_{\Omega^{-}} (\text{div} \, h) E(U, U) \, dx \]

\[ + 2 \int_{\Omega^{-}} h_{i} \partial_{i} U \cdot \Phi \, dx - 2 \int_{\Omega^{-}} (\partial_{p} h_{i}) \left[ (\partial_{i} u_{q}) \tau_{pq}(U) \right] dS \]

\[ + (\partial_{i} \omega_{q}) \mu_{pq}(U) \int_{\Omega^{+}} h_{i} \partial_{i} U \cdot \Phi \, dx \]

(39)

where \( \Phi := L(\partial) U \in [L_{2, \text{loc}}(\Omega^{-})]^{6} \).

**Remark 6.3**
For an arbitrary Lipschitz surface \( S \) we can choose a vector function \( h = (h_{1}, h_{2}, h_{3}) \in [C^{1}(\mathbb{R}^{3})]^{3} \) such that it has a compact support and \( n(x) \cdot h(x) > c_{0} > 0 \) almost everywhere on \( S \) where \( n \) is the exterior normal vector and \( c_{0} \) is some constant (see, e.g. References [19,33]).

**Remark 6.4**
Below we will apply relation (39) to the layer potentials. Let us note that to the asymptotic behaviour of the fundamental matrices at infinity (see Reference [18]) the above-introduced single- and double-layer potentials and their derivatives decay exponentially as \( |x| \to \infty \) if \( \sigma^{2} \) is a sufficiently large negative number.

If \( \sigma = 0 \), then the derivatives of order \( m \) \( (m \geq 0) \) of the single-layer potential vanish as \( O(|x|^{-m-1}) \), while the derivatives of order \( m \) of the double-layer potential decay as \( O(|x|^{-m-2}) \) as \( |x| \to \infty \).

Therefore these potentials belong to the appropriate Sobolev–Slobodetski spaces in \( \Omega^{-} \).

In particular, for \( \sigma = 0 \) all the potentials belong to the well-known Beppo–Levi space (weighted Sobolev space) in \( \Omega^{-} \). As a result we can introduce the corresponding norm as \( \sum_{j=1}^{3} \| \partial_{j} U \|_{[L_{2}(\Omega^{-})]^{6}} \) (for details see, e.g. Reference [34, Chapter 11]).

**6.2. Regularity of the stress vector**
As we have shown, the interior and exterior Dirichlet problems are uniquely solvable in \([H^{1}(\Omega^{\pm})]^{6}\), respectively, and the generalized traces of the corresponding stress vectors \([TU]^{\pm}\) on \( S \) belong to the function space \([H^{-1/2}(S)]^{6}\) (see Theorem 5.3). The following assertion shows that if the Dirichlet data are smoother, the traces of the stress vector on \( S \) become then smoother as well.
Lemma 6.5
Assume that $\Omega^+$ is a Lipschitz domain. Let $U \in [H^1(\Omega^+)]^6$ and the conditions $L(\partial)U =: \Phi \in [L_2(\Omega^+)]^6$ and $[U]^+ \in [H^1(S)]^6$ hold.

Then $[T(\partial, n)U]^+ \in [L_2(S)]^6$ and there exists a positive constant $C$, depending only on the hemitropic constants and Lipschitz parameters of the domain $\Omega^+$, such that

$$\|T(\partial, n)U|^+\|_{[L_2(S)]^6} \leq C\{\|U\|^+_{[H^1(S)]^6} + \|U\|_{[H^1(\Omega^+)]^6} + \|\Phi\|_{[L_2(\Omega^+)]^6}\} \quad (40)$$

Proof
Suppose in the first instance that $U \in [H^2(\Omega^+)]^6$. From (12) and (28) it follows that

$$E(U, U) = E_0(U, U) + \Psi_1(U) \quad (41)$$

where

$$\Psi_1(U) = -4\pi \left(\text{curl } u + \frac{\nu}{\alpha} \text{curl } \omega\right) \cdot \omega + 4\pi |\omega|^2$$

We can represent the differentiation operator $\partial_j$ as a linear combination of the normal and tangential derivatives (see, e.g. Reference [46])

$$\partial_j = n_j \partial_n + \partial, \quad j = 1, 2, 3 \quad (42)$$

where $\partial = (\partial_1, \partial_2, \partial_3)$ denotes Günter’s tangential derivatives which are related to the well-known Stoke’s tangential derivatives

$$\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3) = [n \times \nabla], \quad \mathcal{T}_j = \sum_{p, q=1}^3 \varepsilon_{j pq} n_p \partial_q, \quad j = 1, 2$$

by the relation

$$\partial = -n \times \mathcal{T}$$

Here the symbol $\cdot \times \cdot$ stands for the (formal) cross product.

By virtue of (6) and (17) and applying (42) we get

$$E_0(U, U) = L_0(n) \hat{\partial}_n U \cdot \hat{\partial}_n U + \Psi_2(U) \quad (43)$$

with

$$\Psi_2(U) = \sum_{k, j, p, q} \{a_{k pq} \hat{\partial}_n U_k \partial_p U_q + b_{k pq} \partial_j U_k \partial_p U_q\} \quad (44)$$

where $a_{k pq}$ and $b_{k pq}$ are bounded measurable functions. Note that the matrix $L_0(n)$ (see (6)) is positive definite.

Since $T_0(n, n) = L_0(n)$, it is also evident that (see (2))

$$2h_i \hat{\partial}_i U \cdot T U = 2 \langle n \cdot h \rangle L_0(n) \hat{\partial}_n U \cdot \hat{\partial}_n U + \Psi_3(U) \quad (45)$$

with

$$\Psi_3(U) = \sum_{k, j, p, q, l} \{a'_{k pq l} \hat{\partial}_n U_k \partial_p U_q + b'_{k pq l} \partial_j U_k \partial_p U_q \}
+ \sum_{k, j, p, q} \{c'_{k pq l} U_k \partial_j U_p + d'_{k pq l} U_k \hat{\partial}_n U_p\} h_i \quad (46)$$

where as above $a'_{k pq l}$, $b'_{k pq l}$, $c'_{k pq l}$, and $d'_{k pq l}$ are bounded measurable functions.
Further, with the help of (41), (43)–(46), from the Rellich–Payne–Weinberger identity (34) we deduce

$$\int_S \{(n \cdot h)[L_0(n) \partial_n U \cdot \partial_n U + \Psi_1(U) + \Psi_2(U)]\}^+ dS = 2 \int_S \{(n \cdot h)L_0(n) \partial_n U \cdot \partial_n U + \Psi_1(U)\}^+ dS + \int_{\Omega^+} (\text{div } h)E(U, U) dx$$

$$-2 \int_{\Omega^+} h \partial_1 U \cdot \Phi \, dx - 2 \int_{\Omega^+} (\partial_p h_1)(\partial_j \mu_q) \tau_{pq}(U) + (\partial_j \omega_q) \mu_{pq}(U) \, dx$$

Whence due to Remark 6.3 and positive definiteness of the matrix $L_0(n)$ we arrive at the inequality

$$\|\{\partial_n U\}^+\|_{[L_2(S)]^p} \leq C_1 \{\|U\|_{[H^1(S)]^p} + \|U\|_{[H^1(\Omega^+)]^p} + \|\Phi\|_{[L_2(\Omega^+)]^p}\}$$

Since

$$\|\{TU\}^+\|_{[L_2(S)]^p} \leq C' \{\|U\|_{[H^1(S)]^p} + \|\partial_n U\|_{[L_2(S)]^p}\}$$

inequality (40) follows.

The proof for the case $U \in [H^1(\Omega^+)]^p$ with $L(\partial)U =: \Phi \in [L_2(\Omega^+)]^p$ and $[U]^+ \in [H^1(S)]^p$ can be carried out by the standard limiting procedure developed, e.g. in Reference [33] (see also the proof of Theorem 4.24 in Reference [25]).

**Corollary 6.6**

Assume that $\Omega^-$ is a Lipschitz domain. Let $U \in [H^1_0(\Omega^-)]^p$ and the conditions $L(\partial)U =: \Phi \in [L_2_0(\Omega^-)]^p$ and $[U]^+ \in [H^1(\Omega^-)]^p$ hold.

Then $[T(\partial, n)U]^+ \in [L_2(S)]^p$ and for a fixed positive number $R$ there is a positive constant $C_R$, depending only on $R$, the hemitropic constants and the Lipschitz parameters of the manifold $S$, such that

$$\|[T(\partial, n)U]^+\|_{[L_2(S)]^p} \leq C_R \{\|[U]^+\|_{[H^1(S)]^p} + \|[U]^+\|_{[H^1(\Omega^-)]^p} + \|[\Phi]\|_{[L_2(\Omega^-)]^p}\}$$

with $\Omega^-_R = \Omega^- \cap B(O, R)$, where $B(O, R)$ is an open ball centred at the origin $O$ and radius $R$, such that $\overline{\Omega^-_R} \subset B(O, R)$.

**Remark 6.7**

If $U$, involved in Lemma 6.5 or Corollary 6.6, solves the equation $L(\partial, \sigma) = 0$ in $\Omega^-$, where $\sigma^2$ is a sufficiently large negative number, then we have

$$\|[T(\partial, n)U]^\pm\|_{[L_2(S)]^p} \leq C \|[U]^\pm\|_{[H^1(S)]^p}$$

due to Corollary 5.5. Here $C$ depends only on the hemitropic constants and the Lipschitz parameters of the manifold $S$. 
7. BASIC RESULTS

On the basis of the results obtained in the previous sections, here we establish the invertibility of the basic integral operators (generated by the single- and double–layer potentials) in various function spaces.

7.1. Properties of the operator $H^{(\sigma)}$

**Theorem 7.1**

Let $S$ be a Lipschitz boundary and $\sigma^2$ be a sufficiently large negative number. Then

$$H^{(\sigma)} : [H^{-1/2+r}(S)]^6 \rightarrow [H^{1/2+r}(S)]^6, \quad -\frac{1}{2} \leq r \leq \frac{1}{2}$$

is an isomorphism.

**Proof**

The mapping property and boundedness of operator (47) is obvious and can be shown by standard arguments (cf. e.g. References [24,25,27,28,45,47]).

Further, we know that $H^{(\sigma)} : [H^{-1/2}(S)]^6 \rightarrow [H^{1/2}(S)]^6$ is an isomorphism due to Theorem 5.4. Therefore

$$H^{(\sigma)} g = f$$

with $f \in [H^1(S)]^6$ is uniquely solvable in the space $[H^{-1/2}(S)]^6$.

Let us show that the solution $g \in [L_2(S)]^6$.

Equation (48) implies that the single-layer potential $U(x) = V^{(\sigma)}(g)(x)$ with the density $g \in [H^{-1/2}(S)]^6$ solves the interior and exterior Dirichlet problems

$$L(\partial, \sigma)U(x) = 0 \quad \text{in } \Omega^\pm$$

$$[U]^\pm = f \quad \text{on } S$$

Since $f \in [H^1(S)]^6$ and $U = V^{(\sigma)}(g) \in [H^1(\Omega^\pm)]^6$, we conclude by Lemma 6.5 and Theorem 5.2 that $[TU]^\pm \in [L_2(S)]^6$. Now from the equality $[TU]^\pm = [+2^{-1}g + \mathcal{H}g]$ the embedding $g = [TU]^- - [TU]^+ \in [L_2(S)]^6$ follows. Therefore

$$H^{(\sigma)} : [H^0(S)]^6 \rightarrow [H^1(S)]^6$$

is an isomorphism.

By duality we get that

$$H^{(\sigma)} : [H^{-1}(S)]^6 \rightarrow [H^0(S)]^6$$

is an isomorphism.

Further, by interpolation we conclude that (47) is an isomorphism for $-\frac{1}{2} < r < \frac{1}{2}$.  \[\Box\]
Remark 7.2
Since $-\mathcal{H}^{(\sigma)}: [H^0(S)]^6 \to [H^0(S)]^6$ is compact and positive we have (for details concerning the spectral properties of operators similar to the above-introduced ones see References [27,28]):

(i) the characteristic numbers $\{\lambda_k\}_{k=1}^{\infty}$ are positive and non-decreasing, have the property $\lim_{k \to \infty} \lambda_k = +\infty$, and the corresponding orthonormal system of eigenvectors $\{\varphi_k\}_{k=1}^{\infty}$ (where $\varphi_k + \lambda_k \mathcal{H}^{(\sigma)} \varphi_k = 0$) is complete;
(ii) for an arbitrary vector function $h \in [H^0(S)]^6$ we have $h(x) = \sum_{k=1}^{\infty} h_k \varphi_k(x)$ with $h_k = (h, \varphi_k)_{[H^0(S)]^6}$, where the series converges in $[H^0(S)]^6$ and $\|h\|^2_{[H^0(S)]^6} = \sum_{k=1}^{\infty} |(h, \varphi_k)|^2$.

Now we can prove the following assertion concerning the method of successive approximation for the integral equation of the first kind (48) (cf. Reference [48]).

Theorem 7.3
Let $S$ be a Lipschitz boundary, $f \in [H^1(S)]^6$, and $\sigma^2$ be a sufficiently large negative number. Then the equation $-\mathcal{H}^{(\sigma)} \varphi = f$ is uniquely solvable and $\varphi \in [H^0(S)]^6$.

The sequence

$$
\begin{align*}
g_n &= g_{n-1} + \lambda [f + \mathcal{H}^{(\sigma)} g_{n-1}] \quad \text{for } n \geq 1 \\
g_0 &= [H^0(S)]^6, \quad 0 < \lambda < 2\lambda_1
\end{align*}
$$

(49)–(50)

where $\lambda_1 > 0$ is the least characteristic number of the operator $-\mathcal{H}^{(\sigma)}$, converges in $[H^0(S)]^6$ to the solution vector $\varphi$.

Proof
Unique solvability in $[H^0(S)]^6$ of Equation (48) follows from Theorem 7.1.

Let us set

$$
\begin{align*}
w_n &:= g_n - \varphi, \quad n \geq 0
\end{align*}
$$

(51)

where $g_n$ is given by (49)–(50). Clearly, $w_n \in [H^0(S)]^6$ due to $g_n \in [H^0(S)]^6$.

Our goal is to show that

$$
\lim_{n \to \infty} \|w_n\|_{[H^0(S)]^6} = 0
$$

(52)

From (51) and (49) we have

$$
w_n = w_{n-1} + \lambda \mathcal{H}^{(\sigma)} w_{n-1}
$$

(53)

Multiply (53) by $\varphi_k$ and apply the self-adjointness of $\mathcal{H}^{(\sigma)}$

$$
\begin{align*}
(w_n, \varphi_k)_{[H^0(S)]^6} &= (w_{n-1}, \varphi_k)_{[H^0(S)]^6} + \lambda (\mathcal{H}^{(\sigma)} w_{n-1}, \varphi_k)_{[H^0(S)]^6} \\
&= (w_{n-1}, \varphi_k)_{[H^0(S)]^6} + \lambda (w_{n-1}, \mathcal{H}^{(\sigma)} \varphi_k)_{[H^0(S)]^6} \\
&= (w_{n-1}, \varphi_k)_{[H^0(S)]^6} - \frac{\lambda}{\lambda_k} (w_{n-1}, \varphi_k)_{[H^0(S)]^6} \\
&= \left[ 1 - \frac{\lambda}{\lambda_k} \right] (w_{n-1}, \varphi_k)_{[H^0(S)]^6} \quad \text{for } n \geq 1
\end{align*}
$$

Whence

\[(w_n, \varphi_k)_{H^0(S)^6} = \left[ 1 - \frac{\hat{\lambda}}{\lambda_k} \right]^n (w_0, \varphi_k)_{H^0(S)^6}\]

where \(w_0 = g_0 - \varphi \in [H^0(S)]^6\).

Denote

\[a_k := (w_0, \varphi_k)_{H^0(S)^6}, \quad k \geq 0\]

By Remark 7.2 we have

\[\|w_0\|_{H^0(S)^6}^2 = \sum_{k=1}^{\infty} |a_k|^2 < +\infty\]  \hspace{1cm} (54)

Note that

\[\left[ 1 - \frac{\hat{\lambda}}{\lambda_k} \right]^2 < 1 \quad \text{for all } k \geq 1\]  \hspace{1cm} (55)

due to inequality (50) and monotonicity of the sequence \(\{\lambda_k\}\).

Let \(\varepsilon > 0\) be an arbitrary positive number. In view of (54) there exists a natural number \(N = N(\varepsilon)\) such that

\[\sum_{k=N+1}^{\infty} \left[ 1 - \frac{\hat{\lambda}}{\lambda_k} \right]^{2n} |a_k|^2 \leq \sum_{k=N+1}^{\infty} |a_k|^2 < \frac{\varepsilon}{2}\]  \hspace{1cm} for all \(n \geq 0\)

On the other hand (for fixed \(N(\varepsilon)\)) there exists a natural number \(K = K(N(\varepsilon))\) such that

\[\sum_{k=1}^{N} \left[ 1 - \frac{\hat{\lambda}}{\lambda_k} \right]^{2n} |a_k|^2 < \frac{\varepsilon}{2}\]  \hspace{1cm} for \(n \geq K\)

due to inequality (55).

Thus, given \(\varepsilon > 0\) there exists an integer \(K = K(\varepsilon)\) such that

\[\|w_n\|_{H^0(S)^6} = \sum_{k=1}^{\infty} |(w_k, \varphi_k)|^2 = \sum_{k=1}^{\infty} \left[ 1 - \frac{\hat{\lambda}}{\lambda_k} \right]^{2n} |a_k|^2 < \varepsilon\]

for \(n > K(\varepsilon)\), which yields (52) and completes the proof. \(\square\)

7.2. Properties of the second kind operators

**Theorem 7.4**

Let \(S\) be a Lipschitz boundary and \(\sigma^2\) be a sufficiently large negative number.

Then the operators

\[\pm 2^{-1}I_6 + K^{(\sigma)}: [H^{1/2}(S)]^6 \rightarrow [H^{1/2}(S)]^6\]

\[\pm 2^{-1}I_6 + K^{(\sigma)}: [H^{-1/2}(S)]^6 \rightarrow [H^{-1/2}(S)]^6\]

are isomorphisms.
Proof
Let us consider the bounded linear operator

\[ A^{(\sigma)} := [H^{(\sigma)}]^{-1}(2^{-1}I_6 + \mathcal{K}^{(\sigma)}): [H^{1/2}(S)]^6 \rightarrow [H^{-1/2}(S)]^6 \]  

Due to relation (33) it follows that \( A^{(\sigma)} \) is self-adjoint

\[ A^{(\sigma)*} = (2^{-1}I_6 + \mathcal{K}^{(\sigma)})[H^{(\sigma)}]^{-1} = A^{(\sigma)} \]

In what follows we show that \(-A^{(\sigma)}\) is positive and bounded below, i.e.

\[ \langle -A^{(\sigma)}g, g \rangle_S \geq c_0 \|g\|_{[H^{1/2}(S)]^6} \quad \text{for all } g \in [H^{1/2}(S)]^6 \]  

Let \( g \in [H^{1/2}(S)]^6 \) and put

\[ U(x) = V^{(\sigma)}([H^{(\sigma)}]^{-1}g)(x) \]  

Due to the mapping properties of the single-layer potential (see Theorems 5.2 and 5.4)

\[ V^{(\sigma)}([H^{(\sigma)}]^{-1}g) \in [H^1(\mathbb{R}^3)]^6 \] since \([H^{(\sigma)}]^{-1}g \in [H^{-1/2}(S)]^6\)

Further we apply Green’s formula to vector (58) and take into account the jump relations for the single-layer potential to get

\[ \int_{\Omega^-} E^{(\sigma)}(U, U) \, dx = -\langle [TU]^{-}, [U]^{-} \rangle_S = -\langle (2^{-1}I_6 + \mathcal{K}^{(\sigma)})[H^{(\sigma)}]^{-1}g, g \rangle_S \]

\[ = \langle -A^{(\sigma)}g, g \rangle_S \geq 0 \]  

If \( \langle -A^{(\sigma)}g, g \rangle_S = 0 \) then

\[ \int_{\Omega^-} E^{(\sigma)}(U, U) \, dx = 0 \]

and consequently \( U(x) = 0 \) in \( \Omega^- \) and \([U]^{-} = g = 0\). Thus, \(-A^{(\sigma)}\) is positive:

\[ \langle -A^{(\sigma)}g, g \rangle_S > 0 \] for all \( g \in [H^{1/2}(S)]^6 \) and \( g \neq 0\). On the other hand,

\[ \|g\|_{[H^{1/2}(S)]^6}^2 = \|U^{-}\|_{[H^{1/2}(S)]^6}^2 \leq c_1 \|U\|_{[H^{1}(\Omega^-)]^6}^2 \]

\[ \leq c_3 \int_{\Omega^-} E^{(\sigma)}(U, U) \, dx = c_3 \langle -A^{(\sigma)}g, g \rangle_S \]

due to the continuity of the trace operator, the Korn’s inequality (see Remark 2.6) and formula (59).

This proves the strict coercivity property (57).

By Lax–Milgram’s theorem it follows then that the equation

\[ A^{(\sigma)}g = f \quad \text{for all } f \in [H^{-1/2}(S)]^6 \]
is uniquely solvable in \([H^{1/2}(S)]^6\) and
\[
\|g\|_{[H^{1/2}(S)]^6} \leq c_4\|f\|_{[H^{-1/2}(S)]^6}, \quad \text{with } c_4 = \text{const} > 0
\]
Thus, operator (56) is an isomorphism.

The equations \(A'(\varphi = f)\) and \((2^{-1}I_6 + \mathcal{K}(\varphi) = \mathcal{H}f)\) for \(f \in [H^{-1/2}(S)]^6\) and \(\varphi \in [H^{1/2}(S)]^6\) are equivalent by Theorem 7.1. Therefore, the operator
\[
2^{-1}I_6 + \mathcal{K}(\varphi) : [H^{1/2}(S)]^6 \to [H^{1/2}(S)]^6
\]
is an isomorphism.

By duality the operator
\[
2^{-1}I_6 + \mathcal{K}(\varphi) : [H^{-1/2}(S)]^6 \to [H^{-1/2}(S)]^6
\]
is also an isomorphism.

The proof for the mutually adjoint operators \(-2^{-1}I_6 + \mathcal{K}(\varphi)\) and \(-2^{-1}I_6 + \mathcal{K}(\varphi)\) may be verbatim performed. \(\square\)

Further, we will investigate the above second kind operators acting in the space \([H^0(S)]^6\). First we will prove the following

**Lemma 7.5**

Assume that \(\Omega^+\) (resp. \(\Omega^-\)) is a Lipschitz domain and \(\sigma^2\) is a sufficiently large negative number. Let \(U \in [H^1(\Omega^+)]^6\) (resp. \(U \in [H^1(\Omega^-)]^6\)) be a solution to the equation \(L(\varphi)U = 0\) and \([TU]^+ \in [H^0(S)]^6\) (resp. \([TU]^- \in [H^0(S)]^6\)) hold.

Then \([U]^+ \in [H^1(S)]^6\) (resp. \([U]^- \in [H^1(S)]^6\)) and, moreover, there exists a positive constant \(C\), depending only on the hemitropic constants and Lipschitz parameters of the domain, such that
\[
\int_S \sum_{p,q=1}^3 \left\{ |[\varphi_{p\Omega q}^+]|^2 + |[\varphi_{p\Omega q}^-]|^2 \right\} \, dS \leq C \| [TU]^+ \|^2_{[H^0(S)]^6} \tag{60}
\]
(resp.
\[
\int_S \sum_{p,q=1}^3 \left\{ |[\varphi_{p\Omega q}^-]|^2 + |[\varphi_{p\Omega q}^+]|^2 \right\} \, dS \leq C \| [TU]^-[\|^2_{[H^0(S)]^6} \tag{61}
\]

**Proof**

In accordance with (14) and Remark 6.3 from (34) we get
\[
\int_S \sum_{p,q=1}^3 \left\{ |[\varphi_{p\Omega q}^-]|^2 + |[\varphi_{p\Omega q}^+]|^2 \right\} \, dS \leq c \{ \| [TU]^+ \|^2_{[H^0(S)]^6} + \| U \|^2_{[H^1(\Omega^+)]^6} + \| \Phi \|^2_{[H^0(\Omega^-)]^6} \} \tag{61}
\]
for an arbitrary \(U \in [H^2(\Omega^+)]^6\) provided \(\Phi := L(\varphi)U \in [H^0(\Omega^+)]^6\).

By standard arguments this inequality can be extended to the case: \(U \in [H^1(\Omega^+)]^6\) and \(\Phi := L(\varphi)U \in [H^0(\Omega^+)]^6\) provided that \([TU]^+ \in [H^0(S)]^6\) (cf. References [25,33]).
Let $U \in [H^1(\Omega^+)]^6$ be a unique (weak) solution to the following boundary value problem:

\begin{align}
L(\partial, \sigma)U &= 0 \quad \text{in } \Omega^+ \quad (62) \\
[TU]^+ &= F \quad \text{with } F \in [H^0(S)]^6 \quad (63)
\end{align}

where $\sigma^2$ is a sufficiently large negative number.

Applying the inequality (61) for this solution we then have (60) since

\[ \|\Phi\|_{[H^0(\Omega^+)]^6}^2 = \rho|\sigma|^2 \|\omega\|_{[H^0(\Omega^+)]^6}^2 + |\mathcal{J}| \sigma^2 \|\omega\|_{[H^0(\Omega^+)]^6}^2 \leq (q + |\mathcal{J}|) \sigma^2 \|U\|_{[H^0(\Omega^+)]^6}^2 \]

and due to Corollary 5.5

\[ \|U\|_{[H^0(\Omega^+)]^6} \leq \|U\|_{[H^1(\Omega^+)]^6}^2 \leq c \|[TU]^+\|_{[H^{-1/2}(S)]^6}^2 \leq c \|[TU]^+\|_{[H^0(S)]^6}^2 \]

The proof for the exterior domain $\Omega^-$ is word-for-word. \hfill \Box

From Lemma 7.5 by Corollary 5.5 and the continuity of the trace operator we derive

**Corollary 7.6**

If conditions of Lemma 7.5 are fulfilled, then

\[ \|[U]^\pm\|_{[H^1(S)]^6} \leq C \|[TU]^\pm\|_{[L_2(S)]^6} \]

Further we prove the following

**Theorem 7.7**

Let $S$ be a Lipschitz boundary and $\sigma^2$ be a sufficiently large negative number. Then operators (66)–(68) are isomorphisms.

Moreover, the operators

\[ \pm 2^{-1} I_6 + \mathcal{K}^{(\sigma)} : [H^1(S)]^6 \to [H^1(S)]^6 \]

are invertible.

**Proof**

From (60) it follows that $L_2$-norms of the boundary values on $S$ of the normal and tangential derivatives of $U$ can be controlled by the $L_2$-norm of the boundary value of the corresponding stress vector $[TU]$.

Taking into account the equality (see (42))

\[ |\nabla U|^2 = |\mathcal{D} U|^2 + |\mathcal{D}_n U|^2 \]

we get from (60)

\[ \|[\mathcal{D} U]^\pm\|_{[H^0(S)]^6}^2 = \sum_{p,q=1}^3 \int_S \left\{ |\mathcal{D}_p U_q|^2 + |\mathcal{D}_p U_q|^2 \right\} dS \leq C \|[TU]^\pm\|_{[H^0(S)]^6}^2 \]

Further, let $U = V^{(\sigma)}(g)$ with $g \in [H^0(S)]^6$. Evidently, $U^\pm = \mathcal{K}^{(\sigma)} g \in [H^1(S)]^6$ and by standard arguments it can be shown that the tangential derivatives of the single-layer potential are continuous (cf. e.g. References [19,24,45])

\[ \mathcal{D}[U]^+ = \mathcal{D}[U]^+ = \mathcal{D}[\mathcal{K}^{(\sigma)} g] \]

Applying Corollaries 7.6 and 6.7, and relations (64) we conclude that for arbitrary $g \in [H^0(S)]^6$ we have

$$
\| [TU]^+ \|_{[H^0(S)]^6} = \|(-2^{-1}I_6 + \mathcal{K}^{(\sigma)}) g \|_{[H^0(S)]^6} \\
\leq c_1 \| U^+ \|_{[H^0(S)]^6} = c_1 \| \mathcal{K}^{(\sigma)} g \|_{[H^1(S)]^6} = c_1 \| U^- \|_{[H^1(S)]^6} \\
\leq c_2 \| [TU]^- \|_{[H^0(S)]^6} = c_2 \| (2^{-1}I_6 + \mathcal{K}^{(\sigma)}) g \|_{[H^0(S)]^6} \\
\leq c_3 \| U^- \|_{[H^1(S)]^6} = c_3 \| \mathcal{K}^{(\sigma)} g \|_{[H^1(S)]^6} = c_3 \| U^+ \|_{[H^1(S)]^6} \\
\leq c_4 \| [TU]^+ \|_{[H^0(S)]^6} = c_4 \|(-2^{-1}I_6 + \mathcal{K}^{(\sigma)}) g \|_{[H^0(S)]^6}
$$

(65)

These inequalities lead to the following imbedding results:

The equation

$$( -2^{-1}I_6 + \mathcal{K}^{(\sigma)}) g = F$$

is uniquely solvable for arbitrary $F \in [H^0(S)]^6$ and the solution $g$ belongs to the space $[H^{-1/2}(S)]^6$ due to Theorem 7.4. The corresponding single-layer potential $U = V^{(\sigma)} g \in [H^1(\Omega^+)]^6$ solves the interior BVP (62)–(63).

Since $[TU]^+ = F \in [H^0(S)]^6$, due to Lemma 7.5 we conclude that $[U]^+ = \mathcal{K}^{(\sigma)} g \in [H^1(S)]^6$. Inequalities (65) then imply $g \in [H^0(S)]^6$. Consequently, the operator

$$( -2^{-1}I_6 + \mathcal{K}^{(\sigma)}): [H^0(S)]^6 \rightarrow [H^0(S)]^6$$

(66)

is an isomorphism.

Quite similarly we can show that

$$2^{-1}I_6 + \mathcal{K}^{(\sigma)}: [H^0(S)]^6 \rightarrow [H^0(S)]^6$$

(67)

is an isomorphism.

Then by duality we get that the operators

$$\pm 2^{-1}I_6 + \mathcal{K}^{(\sigma)*} : [H^0(S)]^6 \rightarrow [H^0(S)]^6$$

(68)

are also isomorphisms.

Thus, the first part of the theorem is proved.

The second part of the theorem is a ready consequence of the first one, Lemma 7.1, and the relation (see the first formula in (33))

$$\pm 2^{-1}I_6 + \mathcal{K}^{(\sigma)*} = \mathcal{K}^{(\sigma)} [\pm 2^{-1}I_6 + \mathcal{K}^{(\sigma)}]^{-1}\mathcal{K}^{(\sigma)}$$

The proof is complete.

Remark 7.8

Let $\mathcal{Q}^{(\sigma)} : X \rightarrow Y$ be one of the operators given by (28) with appropriate function spaces $X$ and $Y$. From the results obtained above for the integral operators with superscript $\sigma$ (where $\sigma^2$ is a sufficiently large negative number), it follows that the integral operators corresponding...
to an arbitrary complex $\sigma$ (in particular, for $\sigma = 0$) are Fredholm with zero index. This is due to the fact that for arbitrary two different complex values $\sigma_1$ and $\sigma_2$ the difference $Q^{(\sigma_1)} - Q^{(\sigma_2)}$ is a compact mapping from $X$ into $Y$. In turn, this is a consequence of the structural property of the fundamental matrix $\Gamma(x, \sigma)$ which states that the principal singular part $\Gamma_0(x)$ does not depend on the parameter $\sigma$ (see Section 3 and (20)).

7.3. Existence results

First of all we recall that a closed Lipschitz surface $S$ satisfies the uniform cone condition and vice-versa [49], i.e. each point $x \in S$ is the vertex of two cones $\gamma^{(\pm)}(x)$ with common axis that are congruent to a fixed cone

$$ \{x = (x_1, x_2, x_3) \in \mathbb{R}^3 : 0 \leq x_3 \leq h, \sqrt{x_1^2 + x_2^2} \leq \varepsilon^*(h - x_3)\}, \quad \varepsilon^* > 0, \ h > 0 $$

and such that all points of these cones except $x$ lie in the respective domains $\gamma^{(\pm)}(x) \subset \Omega^{\pm}$. Usually, these cones $\gamma^{(\pm)}(x)$ are called non-tangential approach regions and are subjected to some regularity conditions described, e.g. in Reference [19, Section 0].

In what follows the boundary values $[\cdot]^{\pm}$ on the surface $S$ are taken in the sense of point-wise non-tangential convergence at almost every point with respect to the surface measure (if not otherwise stated). In particular,

$$ [U(x)]^{\pm} = \lim_{\gamma^{(\pm)}(x) \ni y \rightarrow x, y \in S} U(y) \quad (69) $$

$$ [T(\partial_x, n(x))U(x)]^{\pm} = \lim_{\gamma^{(\pm)}(x) \ni y \rightarrow x, y \in S} T(\partial_y, n(x))U(y) \quad (70) $$

for almost all $x \in S$.

Further, by $\mathcal{N}^{\pm}(U) = (\mathcal{N}^{\pm}(U_1), \mathcal{N}^{\pm}(U_2), \ldots, \mathcal{N}^{\pm}(U_6))$, we denote the non-tangential maximal vector function on $S$ corresponding to a vector $U = (U_1, \ldots, U_6)$:

$$ \mathcal{N}^{\pm}(U_k)(x) = \sup_{y \in \gamma^{(\pm)}(x)} |U_k(y)| \text{ for almost all } x \in S $$

(for details see References [19,21,24,27,28,45]).

With the help of the above-obtained results we can prove the following existence theorems.

In what follows by $C$ we denote a constant which depends on the hemitropic parameters of the elastic body and the Lipschitz characteristics of the surface $\partial \Omega^{\pm} = S$.

**Theorem 7.9**

Let $S$ be a Lipschitz boundary and $\sigma^2$ be a sufficiently large negative number. Then the interior and exterior Dirichlet problems

$$ L(\partial, \sigma)U(x) = 0 \quad \text{in } \Omega^{\pm} \quad (71) $$

$$ [U]^{\pm} = f \quad \text{on } S, f \in [L_2(S)]^6 \quad (72) $$

$$ \mathcal{N}^{\pm}(U) \in [L_2(S)]^6 \quad (73) $$

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are uniquely solvable and the solutions can be represented by the double-layer potentials
\[ U(x) = W^{(\sigma)}(g^{(\pm)})(x), \quad x \in \Omega^\pm, \]
where the density vectors \( g^{(\pm)} \in [L_2(S)]^6 \) solve the integral equations
\[ [\pm 2^{-1} I_6 + \mathcal{H}^{(\sigma)}] g^{(\pm)} = f \quad \text{on } S \]  
(74)

Moreover, \( U = W^{(\sigma)}(g^{(\pm)}) \in [H^{1/2}(\Omega^\pm)]^6 \) and
\[ \|U\|_{[H^{1/2}(\Omega^\pm)]^6} \leq C \|f\|_{[L_2(S)]^6} \]  
(75)

Further, if actually \( f \in [H^1(S)]^6 \), then \( U \) can be represented as a single-layer potential
\[ U = V^{(\sigma)}(h) \text{ in } \Omega^\pm, \quad h \in [L_2(S)]^6 \]
and solves the equation \( \mathcal{H}^{(\sigma)} h = f \), and consequently
\[ U = V^{(\sigma)}(h) \in [H^{3/2}(\Omega^\pm)]^6 \]
\[ \|U\|_{[H^{3/2}(\Omega^\pm)]^6} \leq C \|f\|_{[H^1(S)]^6} \]  
(76)

**Proof**

By Theorem 7.7 we can show the existence results and the representability of solutions in
the form of the double-layer potentials with densities \( g^{(\pm)} \in [L_2(S)]^6 \) satisfying equations (74).
Note that these double-layer potentials do not belong to the spaces \([H^1(\Omega^\pm)]^6\), in general.
Therefore to prove the uniqueness of solutions we have to apply the approach based upon the
application of Green’s function technique and standard limiting procedure approximating the
original Lipschitz domain \( \Omega^\pm \) by \( C^\infty \)-smooth sub-domains \( \Omega_j^\pm \) whose Lipschitz characters are
controlled uniformly in \( j \). Word-for-word arguments of the proof of Theorem 3.1 in Reference
[24] show that the homogeneous version of interior and exterior Dirichlet problems (71)–(73)
have only the trivial solution (see also Lemma 4.1 in Reference [45] and Proposition 6.8 in
Reference [27]). Estimate (75) follows then from relations (71) and (73) (cf. Proposition 4.9
in Reference [50] and Theorem 6.2 in Reference [27]).

The second part of the theorem along with estimate (76) follows from Theorem 7.1
and properties of the single-layer potential \( V^{(\sigma)}(h) \) with density \( h \in [L_2(S)]^6 \) (cf. Theorem 3.1
in Reference [24], Corollaries 5.5 and 5.7 in Reference [51], and also Theorem 6.2 in Reference [27]).

\[ \Box \]

**Theorem 7.10**

Let \( S \) and \( \sigma^2 \) be as in Theorem 7.9. Then the interior and exterior Neumann problems
\[ L(\partial, \sigma) U(x) = 0 \quad \text{in } \Omega^\pm \]  
(77)
\[ [T(\partial, n) U]^{\pm} = F \quad \text{on } S, \quad F \in [L_2(S)]^6 \]  
(78)
\[ \mathcal{N}^{\pm}(\partial_j U) \in [L_2(S)]^6, \quad j = 1, 2, 3 \]  
(79)
are uniquely solvable and the solutions can be represented by the single-layer potentials
\[ U(x) = V^{(\sigma)}(h^{(\pm)})(x), \quad x \in \Omega^\pm, \]
where the density vectors \( h^{(\pm)} \in [L_2(S)]^6 \) solve the integral equations
\[ [\pm 2^{-1} I_6 + \mathcal{H}^{(\sigma)}] h^{(\pm)} = F \quad \text{on } S \]
Moreover, \( U = \nu(\sigma)(h^{(\pm)}) \in [H^{3/2}(\Omega)]^6 \) and

\[
\|U\|_{[H^{3/2}(\Omega)]^6} \leq C \|F\|_{[L_2(S)]^6}
\]

**Proof**
It is quite similar to the proof of Theorem 7.9. \( \square \)

**Remark 7.11**
Due to Remark 7.8 we can show that Theorems 7.9 and 7.10 (with slight modification) remain valid for the interior Dirichlet problem \( (I(0))_F \) and the exterior Neumann problem \( (II(0))_F \) of statics (i.e. for \( \sigma = 0 \)), since the null spaces of the corresponding operators \( 2^{-1}I_6 + \mathcal{X}^{(0)*} \) and \( 2^{-1}I_6 + \mathcal{K}(0) \) are trivial.

Note also that the single-layer potential \( \nu(0)(h^-) \) with \( \sigma = 0 \) belongs to the Beppo–Levi space \( BL(\Omega^-) \) (weighted Sobolev space in \( \Omega^- \); see, e.g. Reference [34, Chapter 4, Part B, Section 1]) and under the conditions of Theorem 7.10 we have \( \nu(0)(h^-) \in [H^{3/2}(\Omega^-)]^6 \cap [BL(\Omega^-)]^6 \).

To prove the existence result for the interior Neumann problem we proceed as follows:

Denote by \( X_{\Omega}\{\chi^{(1)}, \chi^{(2)}, \ldots, \chi^{(6)}\} \) the linear span of vectors of rigid displacements in a region \( \Omega \), where, for definiteness, we assume that

\[
\begin{align*}
\chi^{(1)} &= (0, -x_3, x_2, 1, 0, 0)^\top, & \chi^{(4)} &= (1, 0, 0, 0, 0, 0)^\top \\
\chi^{(2)} &= (x_3, 0, -x_1, 0, 1, 0)^\top, & \chi^{(5)} &= (0, 1, 0, 0, 0, 0)^\top \\
\chi^{(3)} &= (-x_2, x_1, 0, 0, 0, 1)^\top, & \chi^{(6)} &= (0, 0, 1, 0, 0, 0)^\top
\end{align*}
\]

The restriction of the space \( X_{\Omega}\{\chi^{(1)}, \chi^{(2)}, \ldots, \chi^{(6)}\} \) onto the boundary \( S = \partial \Omega \) we denote by \( X_\delta\{\chi^{(1)}, \chi^{(2)}, \ldots, \chi^{(6)}\} \). Clearly the vectors \( \{\chi^{(j)}\}_j \) are linearly independent in both spaces \( X_{\Omega} \) and \( X_\delta \). Moreover, \( L(\partial)\chi^{(j)}(x) = 0 \quad x \in \mathbb{R}^3 \) and the corresponding force stress and couple stress tensors vanish in \( \mathbb{R}^3 \). This implies that \( [T(\partial, n)\chi^{(j)}]_S = 0 \) for arbitrary surface \( S \) and \( j = 1, 6 \).

**Lemma 7.12**
The linear span \( X_\delta\{\chi^{(1)}, \chi^{(2)}, \ldots, \chi^{(6)}\} \) represents the null space of the operator \( -2^{-1}I_6 + \mathcal{X}^{(0)*} \).

**Proof**
It can be checked that each vector \( \chi^{(j)} \) solves the homogeneous differential equation \( L(\partial)\chi^{(j)}(x) = 0 \) in \( \Omega^+ \) and \( T(\partial, n)\chi^{(j)}(x) = 0 \) on \( S \). Therefore by the general integral representation formula (see (23) with \( \sigma = 0 \)) we establish that the vectors \( \chi^{(j)} \in X_\delta \) solve the homogeneous integral equation

\[
[-2^{-1}I_6 + \mathcal{X}^{(0)*}]\chi^{(j)} = 0, \quad j = 1, 6
\]

Further it can be shown that the dimension of the null space of the adjoint operator \( -2^{-1}I_6 + \mathcal{X}^{(0)} \) equals to 6, since the corresponding linearly independent solutions of the adjoint homogeneous equation are related to the linearly independent solutions of Problem \( (II(0))_\delta \), i.e. to the vectors of the space \( X_{\Omega} \).

On the other hand, the index of the operator in question is zero (due to Theorem 6.3(ii)), which completes the proof. \( \square \)
The interior Neumann problems (77)–(79) for $\sigma = 0$ is solvable if and only if
\[
\int_S F \cdot \chi^{(j)} \, dS = 0, \quad j = 1, 6
\] (80)
and solutions can be represented by the single-layer potential $U(x) = V^{(0)}(h)(x), x \in \Omega^+$, where the density vector $h \in [L_2(S)]^6$ solves the integral equation
\[
[-2^{-1}I_6 + \mathcal{H}^{(0)}] h = F \quad \text{on } S
\] (81)
The solution $U = V^{(0)}(h) \in [H^{3/2}(\Omega^+)]^6$ is defined modulo a rigid displacement $\chi \in X_{\Omega^+}$, while the force stress and the couple stress tensors are determined uniquely.

**Proof**
The proof is standard. The necessity of conditions (80) follows from Green’s formula (8) with $U'(x) = \chi^{(j)}(x)$. The sufficiency follows from Lemma 7.12 if we look for a solution in the form of a single-layer potential $U(x) = V^{(0)}(h)(x), x \in \Omega^+$. In turn this yields the required embedding result.

Note that the homogeneous version of Equation (81) have 6 linearly independent solutions $\{h^{(k)}\}_{k=1}^6$ and the corresponding single-layer potentials represent rigid displacements in $\Omega^+$, i.e. $V^{(0)}(h^{(k)}) \in X_{\Omega^+}, k = 1, 6$.

**Theorem 7.14**
The exterior Dirichlet problem (71)–(73) for $\sigma = 0$ is uniquely solvable and the solution can be represented by a linear combination of the single- and double-layer potentials $U(x) = W^{(0)}(g)(x) + V^{(0)}(g)(x), x \in \Omega^-$, where the density vector $g \in [L_2(S)]^6$ solves the uniquely solvable integral equation
\[
[-2^{-1}I_6 + \mathcal{H}^{(0)} + \mathcal{K}^{(0)}] g = f \quad \text{on } S
\]
Moreover, $U = W^{(0)}(g) + V^{(0)}(g) \in [H^{1/2}(\Omega^-)]^6$.

Further, if actually $f \in [H^1(S)]^6$, then $U$ can be represented as a single-layer potential $U = V^{(0)}(h)$, where $h \in [L_2(S)]^6$ solves the equation $\mathcal{K}^{(0)} h = f$, and consequently $U = V^{(0)}(h) \in [H^{3/2}_{\text{loc}}(\Omega^-)]^6$.

**Proof**
First let us show that the operator
\[
-2^{-1}I_6 + \mathcal{H}^{(0)*} + \mathcal{K}^{(0)} : [L_2(S)]^6 \to [L_2(S)]^6
\] (82)
is an isomorphism.

Since $\mathcal{H}^{(0)} : [L_2(S)]^6 \to [L_2(S)]^6$ is a compact perturbation and the index of the operator $-2^{-1}I_6 + \mathcal{H}^{(0)*} : [L_2(S)]^6 \to [L_2(S)]^6$ is zero, there remains to prove that the null space of operator (82) is trivial.

To this end let us consider the corresponding adjoint homogeneous equation
\[
[-2^{-1}I_6 + \mathcal{H}^{(0)} + \mathcal{K}^{(0)}] \varphi = 0
\] (83)
with the unknown vector function $\varphi \in [L_2(S)]^6$. This integral equation corresponds to the BVP

$$L(\varphi)U(x) = 0 \quad \text{in } \Omega^+$$

$$[T(\varphi, n)U]^+ + [U]^+ = 0 \quad \text{on } S$$

$$\mathcal{N}^+(\varphi, j)U \in [L_2(S)]^6, \quad j = 1, 2, 3$$

provided $U(x) = V^{(0)}(\varphi)(x), x \in \Omega^+$. Since $\varphi \in [L_2(S)]^6$, we conclude that $U = V^{(0)}(\varphi) \in [H^{1,2}(\Omega^+)]^6$. Further, using Green’s formulae we can show that the BVP (84)–(86) has only the trivial solution, which in turn implies $U(x) = 0$ in $\mathbb{R}^3$. Whence $\varphi = 0$ on $S$ follows.

Thus Equation (83) has only the trivial solution, which yields that operator (82) is injective. Therefore (82) is an isomorphism.

If in addition $f \in [H^1(S)]^6$, then the conclusion of the theorem follows since $U$ can be represented as a single-layer potential $U = V^{(0)}(\psi)$, where $\psi \in [L_2(S)]^6$ solves the equation

$$\mathcal{H}^{(0)} \psi = f.$$

\[\square\]

REFERENCES


