Travel-time sensitivity kernels in ocean acoustic tomography

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Wave-theoretic ocean acoustic propagation modeling is combined with the peak arrival approach for tomographic travel-time observables to derive the sensitivity kernel of travel times with respect to sound-speed variations. This is the Born–Fréchet kernel relating the three-dimensional spatial distribution of sound-speed variations with the induced travel-time variations. The derivation is based on the first Born approximation of the Green’s function. The application of the travel-time sensitivity kernel to an ocean acoustic waveguide gives a picture close to the ray-theoretic one in the case of high frequencies. However, in the low-frequency case, of interest in ocean acoustic tomography, for example, there are significant deviations. Low-frequency travel times are sensitive to sound-speed changes in Fresnel-zone-scale areas surrounding the eigenrays, but not on the eigenrays themselves, where the sensitivity is zero. Further, there are areas of positive sensitivity, where, e.g., a sound-speed increase results in an increase of arrival times, i.e., a further delay of arrivals, in contrast with the common expectation. These findings are confirmed by forward acoustic predictions from a coupled-mode code. © 2004 Acoustical Society of America.

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I. INTRODUCTION

Ocean acoustic tomography was introduced by Munk and Wunsch, as a remote-sensing technique for large-scale monitoring of the ocean interior using low-frequency sound. Measuring the travel/arrival times of pulsed acoustic signals propagating from a source to a distant receiver through the water mass over a multitude of different paths, and exploiting the knowledge about how travel times are affected by the sound-speed (temperature) distribution in the water, the latter can be obtained by inversion.

Ray-theoretic modeling has been the most common approach in ocean acoustic tomography, so arrival times are modeled as travel times along eigenrays connecting the source and receiver (ray arrivals). Their variations are associated with variations of the sound speed along the geometric ray paths, forming the basis of the corresponding inversion scheme (ray tomography). Geometric ray theory is simple to apply and also provides a kernel with which to invert from travel time perturbation to ocean perturbation. Nevertheless, it is a high-frequency asymptotic approximation and suffers from limitations in low-frequency applications. Although the underlying acoustic propagation is linear, geometric ray paths are solutions of the nonlinear eikonal equation, and can be chaotic at long ranges in realistic mediums, which makes them hard to produce and difficult to interpret. Acousticians have long recognized that the infinitely thin geometric ray paths are a limiting case, and the true ray sampling is spread over a region comparable to the Fresnel zone. In an attempt to improve ray-theoretic predictions, Bowlin gave wave-theoretic formulas for ray tube widths, based on single scattering theory. As an alternative to ray-theoretic modeling of travel times, wave-theoretic approaches have been proposed, such as the modal arrival and the peak arrival approach.

The present work applies wave-theoretic modeling to obtain the three-dimensional (3-D) sensitivity kernel of travel-time observables for finite-frequency transmissions in ocean acoustic tomography. The travel-time sensitivity kernel (TSK) is the Born–Fréchet kernel of the linear (first-order) integral representation of travel-time variations in terms of the spatial distribution of sound-speed variation. It represents the 3-D spatial distribution of sensitivity of arrival times to sound-speed variations. In the ray approximation this distribution degenerates into a weakly varying sensitivity along the eigenray corresponding to a particular arrival. This means that ray arrival times are sensitive to medium changes only along the geometric ray paths connecting the source and receiver. The wave-theoretic results approach this approximation only in the high-frequency case, whereas for lower frequencies the ray-theoretic model may be an oversimplification.

In seismic tomography the application of 3-D full-wave theory for travel-time modeling of finite-frequency seismic waves has changed the ray-theoretic picture by revealing that low-frequency travel times are sensitive to sound-speed changes in areas surrounding the eigenrays, but not at all to changes taking place on the eigenrays: the wave-theoretic travel-time sensitivity kernels for seismic travel times attain their maximum at a distance from the eigenrays, within the
first Fresnel zone, whereas the sensitivity is zero on the
eigenrays themselves. Due to their shape, resembling that of
a hollow banana, these kernels are often called banana-
doughnut kernels in the seismic literature. Furthermore, there
are areas of positive sensitivity where a sound-speed increase
slows down (rather than advances) the seismic wave. A
distribution-theoretic analysis of the sensitivity kernel in se-
imic tomography revealed that the unperturbed eigenrays
are contained in the support of the kernel even though its
finite-bandwidth regularization may show a vanishing contrib-
ution on the eigenrays in odd dimensions; in this context
the medium perturbations should be viewed as test functions
on which the kernel acts.

The results from seismic wave propagation motivated
the present study of travel-time sensitivity kernels for ocean
acoustic propagation, in connection with ocean acoustic to-
mography. Even though there are significant differences in
the definition and processing of travel-time observables in se-
imic and ocean acoustic tomography, the results obtained
here exhibit similarities to those reported in the seismic
literature, the more striking ones being the zero sensitivity
on the eigenrays and the emergence of areas of positive
sensitivity at the boundary of the first Fresnel zone about the
eigenrays. Instances of positive sensitivity have been re-
ported before in numerical/forward studies of modal travel
times in range-independent ocean environments. These
findings suggest that, especially in the low-frequency case,
geomeric ray modeling may lead to interpretation errors and
its replacement by full-wave modeling has to be considered.

If the ocean perturbations (or models of ocean perturba-
tions, in the case of tomographic inversions) are smooth,
then the structure in the kernel averages out and the differ-
ence from ray theory will be small. However, for smaller-
scale sound-speed structures such as the seasonal ther-
mocline, or model parametrizations based on sharp-edged
elements such as layers or boxes, the differences will be
significant. For example, in a layered inverse model the ray
sensitivity will be large or zero depending on whether the ray
touches a particular layer; the forward model can therefore
change as the ray path evolves with changing sound speed
structure, complicating the inverse. In these cases the wave-
theoretic kernel can reduce the difficulties or nonlinearities
associated with these models; a smoother kernel should re-
duce nonlinearities in models with vertical or horizontal
scales comparable to Fresnel zone scales.

The contents of this work are organized as follows. In
Sec. II we deal with the Green’s function for ocean acoustic
propagation and its perturbations in the Born approximation.
In Sec. III the problem of travel-time modeling in ocean
acoustic tomography is addressed. Using the notion of peak
arrivals and the Born approximation, an expression is de-

erived for the three-dimensional travel-time sensitivity kernel
of finite-frequency tomographic transmissions. In Sec. IV we
present some numerical results for the travel-time sensitivity
kernel based on normal-mode propagation modeling. Finally,
Sec. V contains a discussion of results and main conclusions
from this work. An explanation for the regimes of zero, posi-
tive, and negative travel-time sensitivity in free space is
given in Appendix A. Further, an expression for the (one-
dimensional) vertical sensitivity of peak arrival times is de-

II. THE GREEN’S FUNCTION

The Green’s function $G(\mathbf{r}|\mathbf{r}_s)$ of an ocean acoustic
waveguide describes the acoustic field of a harmonic point
source of unit strength and satisfies the following inhomoge-
neous Helmholtz equation:

$$\nabla^2 + \frac{\omega^2}{c^2(\mathbf{r})} G(\mathbf{r}|\mathbf{r}_s) = -\delta(\mathbf{r} - \mathbf{r}_s),$$  \hspace{1cm} (1)

where $\mathbf{r}$ is the space vector, $\omega$ the circular frequency of
the source, and $\mathbf{r}_s$ its location, $c(\mathbf{r})$ the sound-speed distribution,
and $\delta$ the Dirac $\delta$ function. The equation above is supple-
mented by boundary and interface conditions according to
which $G$ vanishes at the sea surface whereas pressure and
normal velocity are continuous across interfaces, as well as
by a radiation condition according to which the field decays
away from the source and describes an outgoing wave.

If $S(\mathbf{r})$ is an arbitrary source distribution the induced
acoustic field $P(\mathbf{r})$ satisfying the inhomogeneous Helmholtz
equation,

$$\left[\nabla^2 + \frac{\omega^2}{c^2(\mathbf{r})}\right] P(\mathbf{r}) = S(\mathbf{r}),$$  \hspace{1cm} (2)

and the boundary/interface/radiation conditions mentioned
above, can be represented through the Green’s function by the
integral

$$P(\mathbf{r}) = -\int \int \int_V G(\mathbf{r}|\mathbf{r}') S(\mathbf{r}') dV(\mathbf{r}'),$$  \hspace{1cm} (3)

i.e., it is a superposition of the acoustic fields of point
sources distributed over the support of $S(\mathbf{r})$.

A. Born approximation

A perturbation of the sound-speed distribution in Eq. (1)
by $\Delta c$ will cause a perturbation in the Green’s function by
$\Delta G$. The perturbed Green’s function $G+\Delta G$ will satisfy the equation

$$\left[\nabla^2 + \frac{\omega^2}{c^2(\mathbf{r})}\right] (G(\mathbf{r}|\mathbf{r}_s) + \Delta G(\mathbf{r}|\mathbf{r}_s))$$

$$= -\delta(\mathbf{r} - \mathbf{r}_s).$$  \hspace{1cm} (4)

By subtracting Eq. (4) from Eq. (1) and adding the term
$\omega^2 \Delta G/c^2$ to both sides, the following equation is obtained:

$$\left[\nabla^2 + \frac{\omega^2}{c^2(\mathbf{r})}\right] \Delta G(\mathbf{r}|\mathbf{r}_s) = -\frac{\omega^2}{c^2(\mathbf{r})} \frac{\omega^2}{[c(\mathbf{r})+\Delta c(\mathbf{r})]^2} \times [G(\mathbf{r}|\mathbf{r}_s) + \Delta G(\mathbf{r}|\mathbf{r}_s)].$$  \hspace{1cm} (5)

The perturbation $\Delta G$ satisfies the same boundary/inter-
face/radiation conditions as the unperturbed Green’s
function. In this connection, the integral representation (3)
can be used by considering the right-hand side of Eq. (5), as
a function of $\mathbf{r}$, to be the source term.
\[
\Delta G(r|r_s) = \int \int V G(r'|r) \left[ \frac{\omega^2}{[c(r') + \Delta c(r')]^2} \right] 
- \frac{\omega^2}{c^2(r')} [G(r'|r_s) + \Delta G(r'|r_s)] dV(r').
\]

(6)

By retaining terms of first order, the following approximation for the perturbation of the Green’s function is obtained:

\[
\Delta G(r|r_s) = -2\omega^2 \int \int V G(r'|r_s)G(r|r') \times \frac{\Delta c(r')}{c^3(r')} dV(r').
\]

(7)

This is called the first Born approximation. It is also called the single- or weak-scattering approximation, or wave-field linearization and is widely used in connection with scattering problems. It is a linear relation between the perturbation in the sound-speed distribution and the induced perturbation in the Green’s function.

The Green’s function above is defined in the frequency domain—the circular frequency \(\omega\) enters the Helmholtz equation as a parameter. Further, the Green’s function depends on the sound-speed distribution that also enters the Helmholtz equation. In order to explicitly describe these dependencies, \(G\) is written in the following as \(G(r|r_s;\omega;c)\).

III. TRAVEL-TIME MODELING

In ocean acoustic tomography the acoustic field of an impulsive (broadband) point source is recorded at a distant receiver in the time domain. This can be expressed through the inverse Fourier transform in terms of the source signal \(P_s(\omega)\) in the frequency domain and the frequency–domain Green’s function evaluated at the receiver’s location \(r = r_s\),

\[
p_a(t;c) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(r_s|r_s;\omega;c)P_s(\omega)e^{i\omega t} d\omega.
\]

(8)

Due to multipath propagation, the pressure magnitude at the receiver in the time domain \(a(t;c) = |p_a(t;c)|\) consists, in general, of a number of peaks, the acoustic arrivals, whose shape and temporal locations (arrival times) \(\tau_i, i = 1,2,...,I\), are dependent on the sound-speed distribution within the water column. The function \(a(t;c)\) is called the arrival pattern. For most ocean tomography experiments to date, small-scale sound speed variability makes the magnitudes of each peak rapidly variable in time, while the corresponding arrival times \(\tau_i\) are stable and reliable characteristics of the arrival pattern. Arrival times thus constitute the basic set of observables in most ocean tomography experiments and their perturbations are used for the retrieval of the underlying sound-speed perturbations.

This is a simplified, yet sufficient, description of the modeling problem in ocean acoustic tomography, neglecting signal processing issues of importance for the experimental implementation, but not essential for modeling purposes. In practice, it is the autocorrelation of the emitted signal that is impulsive, not the signal itself, and the arrival pattern results from the cross-correlation of the received signal with a replica of the emitted signal (matched filter), equivalent expressions to those presented here hold for the corresponding correlated quantities in the time-lag domain.

For the wave-theoretic modeling of arrival times the notion of peak arrivals is used. Peak arrivals are defined as the significant local maxima of the arrival pattern. The peak-arrival times are the time instants \(\tau_i, i = 1,2,...,I\), corresponding to these maxima,

\[
\dot{a}(\tau_i;c) = 0,
\]

(9)

where the overdot denotes differentiation with respect to time. Expressing the complex pressure \(p_a\) in terms of its real and imaginary parts \(p_a(t;c) = v(t;c) + jw(t;c)\), the definition above can be alternatively expressed as

\[
v(\tau_i;c)\dot{v}(\tau_i;c) + w(\tau_i;c)\dot{w}(\tau_i;c) = 0.
\]

(10)

The definition of peak arrivals has its origin in the procedure followed in experimental practice for obtaining arrivals and arrival times and allows for any kind of modeling approach, either geometric or wave-theoretic. Since the arrival pattern depends on the sound-speed distribution, the peak-arrival times do so as well, i.e., \(\tau_i = \tau_i(c)\).

A perturbation \(\Delta c\) of the sound-speed distribution will cause a variation \(\Delta G\) in the Green’s function, expressed to the first order by Eq. (7), which, in turn, will cause a variation \(\Delta p_a\) in the acoustic pressure at the receiver in the time domain,

\[
\Delta p_a(t;c;\Delta c) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \Delta G(r|r_s;\omega;c;\Delta c) \times P_s(\omega)e^{i\omega t} d\omega.
\]

(11)

The pressure perturbation will produce a perturbation of the arrival pattern and finally a perturbation \(\Delta \tau_i\) of the peak arrival times. According to their definition (9), the perturbed peak arrival times will satisfy the equation

\[
\dot{a}(\tau_i + \Delta \tau_i;c + \Delta c) = 0.
\]

(12)

Using a first-order Taylor expansion with respect to time about \(\tau_i\), the equation above can be written in the form

\[
\dot{a}(\tau_i + \Delta c) + \dot{a}(\tau_i;c + \Delta c)\Delta \tau_i = 0.
\]

(13)

Expressing the pressure perturbation \(\Delta p_a\) in terms of its real and imaginary parts, \(\Delta p_a(t;c;\Delta c) = \Delta v(t;c;\Delta c) + j\Delta w(t;c;\Delta c)\), and taking into account the relations \(a(t;c;\Delta c) = \sqrt{v^2 + w^2}\) and \(a(t;c + \Delta c) = \sqrt{(v + \Delta v)^2 + (w + \Delta w)^2}\), as well as Eq. (10), holding at the unperturbed arrival time \(\tau_i\), the following expression can be derived from (13) for the perturbation of the peak arrival time,

\[
\Delta \tau_i = -\frac{\dot{\Delta} v_j + v_j \Delta \dot{v}_j + \dot{w}_j \Delta w_j + w_j \Delta \dot{w}_j}{\dot{v}_j^2 + v_j^2 \dot{w}_j + w_j^2 \dot{w}_j}.
\]

(14)

The index \(i\) denotes that all quantities on the right-hand side of Eq. (14) are evaluated at the background (unperturbed) arrival time \(\tau_i\), i.e., for a particular peak they have fixed values.
A. Travel-time sensitivity kernel

Equation (14) relates the arrival-time perturbation $\Delta \tau_i$ with the perturbation $\Delta \rho_i$ of the acoustic pressure at the receiver in the time domain. The latter is related to the perturbation $\Delta G$ of the Green’s function through Eq. (11). Finally, $\Delta G$ is related to the perturbation $\Delta c$ in the sound-speed distribution through the first Born approximation (7). By combining these, an integral relation can be derived, expressing the arrival-time perturbation $\Delta \tau_i$ in terms of the underlying sound-speed variation $\Delta c$.

$$\Delta \tau_i = \int \int V \Delta c(r') K_i(r'|r_i; r_r; c) dV(r').$$

(15)

where $K_i(r'|r_i; r_r; c)$ is the sensitivity kernel for the $i$th peak arrival and is given by

$$K_i(r'|r_i; r_r; c) = \Re \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{v_i + jw_i}{b_i} \right. \times Q(r'|r_i; r_r; c) e^{i\omega \tau_i} d\omega$$

$$+ \Im \left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\dot{v}_i + j\dot{w}_i}{b_i} Q(r'|r_i; r_r; c) e^{i\omega \tau_i} d\omega \right\},$$

(16)

where $\Re$ and $\Im$ denote real and imaginary parts, respectively, $b_i = v_i^2 + v_r^2 + w_i^2 + w_r^2$ is one-half the second time derivative of the squared arrival pattern $a^2 = v^2 + w^2$ at the background peak arrival time, and

$$Q(r'|r_i; r_r; c) = G(r'|r_i; r_r; c) G(r_i|r'; r_r; c)$$

$$\times \frac{2\omega^2 P_+(\omega)}{c^3(r')}. \tag{17}$$

The linear operator $\int\int \int (K_i \cdot dV$, Eq. (15), resulting from the first Born approximation, is the Fréchet derivative of the travel time $\tau_i$ with respect to the sound-speed distribution. In this connection, the travel-time sensitivity kernel $K_i$ is a Born–Fréchet kernel relating sound-speed and travel-time perturbations. By rearranging terms, the kernel $K_i(r'|r_i; r_r; c)$ can be alternatively written as follows:

$$K_i(r'|r_i; r_r; c) = \Re \left\{ \frac{v_i - jw_i}{2\pi b_i} \int_{-\infty}^{\infty} jw Q(r'|r_i; r_r; c) \right. \times e^{i\omega \tau_i} d\omega$$

$$+ \left\{ \frac{v_i - jw_i}{2\pi b_i} \int_{-\infty}^{\infty} \omega Q(r'|r_i; r_r; c) e^{i\omega \tau_i} d\omega \right\}. \tag{18}$$

On the basis of Eq. (15), the kernel $K_i(r'|r_i; r_r; c)$ gives a quantitative description of the sensitivity of the peak arrival time $\tau_i$ to sound-speed changes at any (three-dimensional) location $r'$ within the acoustic waveguide. A positive value for $K_i$ at a particular location indicates that a sound-speed increase at that location will cause an increase in travel time. A negative value, on the other hand, indicates that a sound-speed increase will cause a decrease in travel time. Finally, a zero indicates that a change in sound speed will have no effect on the travel time.

B. Normal-mode representation

In the case of a horizontally stratified (range-independent) background environment, normal-mode theory can be used to represent the background Green’s function $G$, which is axisymmetric about the vertical axis through the source. A three-dimensional cylindrical coordinate system $(r, z, \phi)$ is used with its origin at the sea surface and the source located on the vertical $z$ axis (positive downward) at depth $z_0 = z_s$. The background sound speed will be a function of depth only: $c = c(z)$. If the source is harmonic, with unit strength, circular frequency $\omega$ and time dependence $e^{i\omega t}$, the farfield expression for the Green’s function at $(r, z)$ can be expressed in the form

$$G(r, z|z_s; \omega; c) = e^{-j\pi\frac{n}{2}} \sum_{n=1}^{M} \frac{u_n(z_s)u_n(z)}{\sqrt{k_n r}} e^{-j k_n r}, \tag{19}$$

where $\rho$ is the water density and $k_n$ and $u_n$, $n = 1, ..., M$, are the real eigenvalues and the corresponding eigenfunctions (propagating modes) of the vertical Sturm–Liouville problem:

$$\frac{d^2 u_n(z)}{dz^2} + \frac{\omega^2}{c^2(z)} u_n(z) = k_n^2 u_n(z), \tag{20}$$

supplemented by the conditions that $u_n = 0$ at the sea surface ($z = 0$), $u_n$ and $\rho^{-1} du_n/dz$ are continuous across the interfaces, and $u_n$ and $du_n/dz$ are vanishing as $z \to \infty$. By substituting (19) into (17), the quantity $Q$ can be expressed as follows:

$$Q(r, z, \phi|z_s; R, z_r; \omega; c)$$

$$= \frac{\omega^2 P_+(\omega)}{4\pi \rho^2 c^3(z)} \sum_{n=1}^{M} \frac{u_m(z_s)u_m(z)}{\sqrt{k_m r}} e^{-j k_m r} \times \sum_{n=1}^{M} \frac{u_n(z)u_n(z_r)}{\sqrt{k_n r + R^2 - 2R \cos \phi}} e^{-j k_n \sqrt{R^2 + R^2 - 2R \cos \phi}}. \tag{21}$$

where $z_s$ and $z_r$ are the source and receiver depths, respectively, and $R$ is the horizontal distance between the source and receiver. The expression (21) can be used for the calculation of the kernel $K_i$ at any location $(r, z, \phi)$.

IV. NUMERICAL RESULTS

In this section we present numerical results for the travel-time sensitivity kernel, assuming three-dimensional (3-D) perturbations of a horizontally stratified background environment. In this connection, the normal-mode representations can be applied. The water depth is 2500 m. The background sound-speed profile in the water is assumed to be
linear, with 1503 m/s at the sea surface and 1547 m/s at the bottom. Both the source and receiver are placed at 150 m depth with horizontal distance 52 km. This configuration is motivated by the Thetis\textsuperscript{10} tomography experiment that took place in the Gulf of Lions (western Mediterranean sea) in winter 1991–92. An absorbing bottom\textsuperscript{3,31} is assumed, filtering out the bottom-interacting part of the acoustic energy. The upward refracting sound speed profile also reduces the influence of the relatively shallow bottom.

Figure 1 shows ray-theoretic results for the example configuration. In particular, the ray intensity, launch angle, and turning depth versus arrival time, as well as the geometry of the corresponding eigenrays are shown in this figure from top to bottom.\textsuperscript{12} There are 15 non-bottom-interacting eigenrays spanning the upper 1100 m of the water column. All of them are subject to surface reflections, due to the upward refracting sound-speed profile. The eigenray with the steepest one and has the earliest arrival time. The eigenrays ± 4 are slightly shallower and arrive about 30 ms later, with identical arrival times, forming the central arrival of a ray group (triplet). The eigenray ± 5 corresponds to the late arrival of the first triplet. Later arrivals are due to gradually shallower eigenrays. The shallowest ray, with a lower turning depth just above 150 m is ±10, corresponding to the last arrival.

The top of Fig. 2 shows the wave-theoretic prediction for the arrival pattern, assuming that the emitted signal is a Gaussian pulse with central frequency 400 Hz and bandwidth 60 Hz (3-dB bandwidth). The broadband calculation in the frequency domain was carried out with a normal-mode code at 191 frequencies, from 305 to 495 Hz with frequency step 1 Hz, and a standard FFT was applied to obtain time-domain results. The first five arrivals in Fig. 2 can be directly associated with individual ray arrivals in Fig. 1, whereas the last peak corresponds to the group of the six latest ray arrivals. In the lower panels of Fig. 2, the travel-time sensitivity kernel (TSK) is presented for the first three arrivals (peaks 1, 2, and 3). In particular, a section of the 3-D TSK is shown in the vertical plane through the source/receiver—the TSK has mirror symmetry about this plane. The Green’s function underlying the TSK calculations was evaluated at 191 frequencies, as above, with a spatial resolution of 5 m in depth and 260 m in range. A moving average window, 100 m in the vertical and 520 m in the horizontal, was applied to the graphical presentation in order to suppress short-scale oscillations due to mode interference.

Peaks 1 and 3 correspond to eigenrays ± 3 and ± 5, respectively (cf. Fig. 1), which are also shown in Fig. 2 (dashed lines). The TSK reproduces the eigenray geometry and also provides quantitative information about the sensitivity of the travel times to sound-speed changes anywhere in the medium. The maximum sensitivity is near the eigenray paths where the TSK takes negative values. This means that a sound-speed decrease along the eigenray will result in a travel-time increase, as expected. The oscillatory behavior of the TSK away from the eigenrays is associated with Fresnel-zone behavior and is expected to cancel out in the integral (15), provided that the sound-speed perturbation is a smooth function of $r$; in the limit $\omega \rightarrow \infty$, the wave-theoretic travel-time sensitivity kernel converges to the ray-theoretic one.\textsuperscript{33} Close to the deep turning points, it is observed that the negative sensitivity maximum is not at the core of the high-sensitivity area, i.e., not along the eigenray, but at a distance from it. This spread sensitivity will become more pronounced in the next example where the frequency is reduced. Peak 2 corresponds to two symmetric eigenrays (± 4). The geometry of the two eigenrays is well reproduced by the TSK of peak 2. While the eigenrays are well separated in the upper 600 m layer, there is overlapping close to the turning points. The TSK of an arrival peak made up of two different ray paths is not generally useful for inversion, since in a realistic ocean environment ray intensities will vary, and one or the other of the two peaks may dominate at any time. A vertical array of hydrophones at the receiver could separate the two rays by time-delay beamforming for the different angles, but that is beyond the scope of this work.

The top panel in Fig. 3 presents the arrival pattern of a lower-frequency Gaussian pulse: The central frequency is now 100 Hz while the bandwidth is the same as before (60 Hz). The first 3 peaks marked in the arrival pattern correspond to the first triplet in Figs. 1 and 2. The TSK for each of these three peaks are shown in the lower panels of Fig. 3, along with the corresponding eigenrays (dashed lines). The more striking change from the previous 400-Hz case is that the domain of lower sensitivity of peaks 1 and 3 along the eigenrays has increased drastically in length and width, and in fact, it has become a domain of insensitivity along the eigenray. On the other hand, the (negative) maximum of the

![FIG. 1. Ray-theoretic results for a linear sound-speed profile (bottom left), for source/receiver depth 150 m and distance 52 km. The graphs show (from top to bottom) the eigenray intensities, launch angles, turning depths versus arrival time, and the eigenray geometry.](image-url)
travel-time sensitivity is attained in an annulus surrounding the eigenray with variable diameter reaching 280 m. Further, there are areas of positive sensitivity, where a sound-speed increase will slow down the acoustic wave.

The broadening of the high-sensitivity domain for lower frequencies can be explained in terms of Fresnel zone behavior, and in this sense it was expected, e.g., in the context of the ray-tube approach. However, the domain of low (vanishing) sensitivity in the middle along the eigenray was not, and it was a surprise for the seismic tomography community, where it was first encountered. The picture is more complex in the case of peak 2 due to the overlapping of the two contributing eigenrays. The two boxes marked on the TSK of peak 1 in Fig. 3 denote the locations of test perturbations for verification purposes and will be addressed shortly.

Figure 4 shows a cross-section of the 100-Hz TSK for the first three peaks at ranges 10, 18, and 26 km. The color scales in this figure are relative to the local maxima and are, in general, different from those of Fig. 3. Since the source/receiver configuration is symmetric, the results for 10 km also apply for the range of 42 km. Similarly, the results for 18 km also apply for the range of 34 km. At the range of 10 (42) km, the cross-section shows that the negative sensitivity annulus close to its broadest position has a horizontal extent of about 400 m and a vertical extent of about 280 m. In the center of the annulus, where the eigenrays pass (positions denoted by asterisks), the sensitivity is zero. At the range of 18 (34) km, the TSK collapses in the vertical while preserving its horizontal extent. Finally, at the source–receiver mid-range (26 km), where reflection takes place for peaks 1 and 3 complicated patterns, occur with strong positive and negative sensitivities.

An explanation of the generating mechanism for the regimes of negative/positive/zero travel-time sensitivity in free space is given in Appendix A, in terms of acoustic scattering taking place in the vicinity of an eigenray. The scattered field may cause advancement or the further delay to an acoustic arrival, or have no effect at all, depending on its phase and arrival time at the receiver. The latter are directly associated with the location of the scatterer, thus defining positive, negative, and zero sensitivity regimes. The sensitivity is zero on the eigenray, whereas the negative sensitivity maximum is attained at a distance from the eigenray. The negative-sensitivity core is contained within the first Fresnel zone of radius $R_F$ and is followed by a positive-sensitivity regime at the boundary of the first Fresnel zone. Thus, the characteristic radius for the positive-sensitivity regime is the Fresnel radius $R_F$, whereas for the negative-sensitivity core it is $R_F/\sqrt{2}$.

Assuming propagation in free space, the Fresnel radius as a function of distance $r$ from the source is given by
where $R$ is the source–receiver distance, $f$ is the frequency of propagation, and $c$ is the sound speed. Taking $R = 52,000 \text{ m}$, $c = 1500 \text{ m/s}$, and $f = 100 \text{ Hz}$, the Fresnel radius at the ranges of 10, 18, and 26 km is 348, 420, and 441 m, respectively. The horizontal (cross-range) extent of the high-sensitivity regimes in Fig. 4 are in agreement with these estimates. For example, at the range of 10 km, the horizontal distance of the positive-sensitivity area from the corresponding eigenray for peak 3 at is about 350 m, whereas the outer horizontal semiaxis of the negative-sensitivity core is about 250 m. The presence of stratification causes significant deformations of the TSK structure in the vertical, from deviations from the circular symmetry to full collapse of the negative-sensitivity regime.

A test of the observed positive/negative travel-time sensitivity in the 100-Hz case was carried out to check the TSK calculations. For this purpose sound-speed perturbations in two areas, A and B, of positive and negative sensitivity, respectively, were introduced and the effects on peak 1 were calculated using a coupled-mode code. The areas A and B are identified in Figs. 3 and 4 and their location/dimensions are given in Table I. The perturbations are axisymmetric with respect to the $z$ axis, such that the domains A and B are not limited in the azimuthal, as shown in Fig. 4. In each perturbed realization the sound speed was raised by 4 m/s at the mid-depth of the perturbation area, linearly decreasing toward the unperturbed values at the upper and lower limits of the area. In this connection, the propagation is divided into three range-independent subdomains. One from the source to the starting range of the perturbation area, one covering the perturbation area, and one from its end range to the receiver.

The results from the forward calculations are shown in Fig. 5. The top panel shows the unperturbed state of peak 1, focusing in the area about its maximum (marked by an asterisk in the top panel and by a vertical dashed line in the panels below). The perturbation in case A has resulted in a delay of peak 1 that is compatible with the positive values of the TSK in area A; cf. Figs. 3 and 4. The perturbation in case B where the negative TSK values dominate has the opposite effect of causing an advancement of peak 1. Note that both areas A and B lie well off the eigenray associated with peak 1. The effect in case A is stronger than in case B. This is attributed to the different azimuthal extent of the high sensitivity areas in the two cases, as shown in Fig. 4. Thus, the forward calculations verify the TSK behavior observed in the previous figures. It is noted that the sound-speed perturbations in both cases A and B reduce the amplitude of peak 1.

Figure 6 shows the TSK for the last peak of the arrival.
patterns of Figs. 2 and 3 for the two cases of central frequencies 400 and 100 Hz. In the ray-theoretic results the last arrival is \(6\) with deep turning depth just above 150 m. In this connection, a smaller moving average window, 40 m in the vertical and 260 m in the horizontal, was applied for the presentation in Fig. 6. The ten turning points of ray arrival \(6\) can be recognized in the geometry of the negative TSK areas in Fig. 6, especially in the 400-Hz case; the ray arrival \(6\) is associated with two symmetric eigenrays also shown in Fig. 6 dashed lines. Still there are structural differences in the TSK patterns as the frequency changes. This suggests that the influence coefficients will be different for different frequency characteristics of the emitted signal (pulse shape, central frequency, bandwidth). The frequency effects on the influence coefficients of late arrivals have been studied in a previous work.37

Figure 7 shows the range-average travel-time sensitivity kernel, i.e., the vertical sensitivity, for peaks 1, 2, and 3 in the 100- and 400-Hz cases. The solid lines represent the wave-theoretic kernels (after the application of a 100-m moving average window), whereas the dashed lines are the ones calculated from ray theory. The wave-theoretic vertical sensitivity kernel was obtained from perturbation analysis of the range-independent problem, as discussed in an earlier paper,11 rather than by integrating the 3-D TSK. The range-independent perturbation analysis is presented in Appendix B. From Fig. 7, it is seen that the oscillations of the full-wave sensitivity persist in the range average, even with vertical smoothing, and it is clear that the sharp edge of the geometric kernel at the lower turning depth is blurred. At 400 Hz (upper panel), the smoothing is not severe, but at 100 Hz (lower panel), the turning point sensitivity is spread by about 200 m, which is significant for internal wave perturbations or practical inverse models.

V. DISCUSSION AND CONCLUSIONS

According to geometric ray theory, travel times are only sensitive to sound-speed changes on the corresponding eigenrays connecting the source and receiver. In ray tomography travel times are considered as integral measures of the reciprocal sound-speed (slowness) along the geometric eigenrays, and ray inversion methods infer sound-speed perturbations from their effects on these line integrals. Nevertheless, ray theory is a high-frequency asymptotic approximation, and a question arises if the geometric sensitivity kernel remains valid for the low frequencies used in ocean acoustic tomography experiments.

To address this question a wave-theoretic modeling approach for acoustic propagation, allowing for the treatment of finite-frequency effects, was combined here with a general modeling approach for travel-time observables, the peak arrival approach, that reflects the procedure followed in experimental practice for obtaining arrivals and arrival times. In this way a general wave-theoretic expression for the 3-D travel-time sensitivity kernel was obtained, which reveals in a quantitative manner the effect that sound-speed perturbations and their spatial distribution have on finite-frequency travel times.

In the case of higher frequencies, the wave-theoretic calculations result in a picture of the travel-time sensitivity that is close to the ray picture. The sensitivity kernel is concentrated around the corresponding eigenrays and decays away from them forming an oscillating Fresnel pattern. In the asymptotic high-frequency limit the integral contribution of the kernel is coming from the first Fresnel zone around the eigenray, as can be obtained by applying the stationary-phase approach to the cross-ray integral, reproducing the ray-theoretic sensitivity.16,33
In the low-frequency case, the wave-theoretic results exhibit significant deviations from ray theory: Low-frequency travel times of early arrivals, corresponding to steep rays, are sensitive to sound-speed changes in areas surrounding the eigenrays, but not on the eigenrays themselves. The sensitivity kernels for these arrivals attain their negative maxima at a distance from the eigenrays whereas the sensitivity is zero on the eigenrays. Still, in a distribution-theoretic context, the eigenrays belong to the support of the sensitivity kernel, in the sense that the result of the kernel acting on any finite-support perturbation of the medium about the eigenray will be non-zero in general. Further, there are areas of positive sensitivity, where, e.g., a sound-speed increase results in an increase of arrival times, i.e., a further delay of arrivals. An explanation for the travel-time sensitivity behavior in free space is given in Appendix A. The presence of horizontal stratification suppresses/deforms the TSK structure in the vertical, giving rise to elliptical rather than circular sensitivity patterns about the eigenrays.

Low-frequency travel times exhibit sensitivity to sound-speed changes at distances 100 m or more beyond the turning depth of the corresponding ray. This sensitivity is also present in the range average, where the wave-theoretic vertical sensitivity kernel oscillates around the ray sensitivity, even exhibiting regions of positive sensitivity that cannot be explained by ray theory. In this connection, the use of path-integral methods based on ray theory for evaluating the effects of features, such as internal waves, on travel times may be inaccurate for low acoustic frequencies and high-vertical-mode ocean features. The spreading of the sensitivity past the turning point is due to diffraction, which is partially responsible for the increased vertical spread of time fronts in long-range ocean transmission experiments.

In seismic transmission tomography, the behavior of travel-time sensitivity kernels of compressional and shear waves has been the objective of a number of papers based on the assumption of known source characteristics in the space/time/frequency domains. In seismic transmission tomography, however, the excitation of the medium is due to earthquakes, and the characterization of natural earth-quakes as seismic sources is associated with ambiguities which in turn affect the travel-time sensitivity kernel. This problem does not arise in ocean acoustic tomography, due to the fact that the sources are controlled and their spatial, temporal and frequency characteristics are known.

Travel-time sensitivity kernels offer a method to study the effect of 3-D spatial scales of sound-speed variations on finite-frequency travel-time observables. This is particularly important for the study of ocean propagation in range-dependent media, such as internal-wave fields. In this connection, the sensitivity kernels have been used for the study of wavefront healing, a diffraction phenomenon that affects travel times when the scale of the 3-D variations in the wave speed is comparable with the characteristic wavelength.

The background state for the numerical results presented in the previous section was taken to be range independent. Nevertheless, the present derivation and the expressions for the travel-time sensitivity kernel are general and can apply to range-dependent background environments as well. This could provide a wave-theoretic tool to study finite-frequency travel-time effects of ray and “wave” chaos—an interesting area for future work. A further area of interest would be the study of the TSK behavior for very long range propagation in order to see how the TSK structure is affected by the broadening of the Fresnel zones; the Fresnel radius is proportional to $\sqrt{R}$ and should become very large for propagation over megameter ranges.

The present derivation for the travel-time sensitivity kernel was based on the first Born (weak scattering) approximation. This approximation is known to be reliable in the case of small perturbations. Other methods such as the Rytov approximation provide a better representation of the transmitted (or forward scattered) part of the wave field. The use of the Born approximation in this work is justified by the fact that it is applied for obtaining an expression for the Fréchet derivative of travel times with respect to sound-speed perturbations, and in this connection infinitesimally small perturbations can be assumed.

APPENDIX A: TRAVEL-TIME SENSITIVITY IN FREE SPACE

In this appendix an explanation is given for the zero/ negative/positive sensitivity of finite-frequency travel times to sound-speed perturbations in free space. Let a point source (of circular frequency $\omega$) and a receiver at fixed locations $r_s$ and $r_r$, respectively, in an unbounded medium with constant background sound speed $c$. The corresponding Green’s function at the location of the receiver has the form

$$G_s(\omega) = \frac{e^{-ikR}}{4\pi R} = \frac{e^{-i\omega T}}{4\pi R},$$

(A1)

where $R = |r_s - r_r|$ is the source–receiver distance, $k = \omega/c$ and $T = R/c$.

A scatterer (a volume $dV$ of sound-speed change $\Delta c$) at distance $R_1$ from the source and $R_2$ from the receiver will cause a perturbation to the Green’s function, which, according to the first Born approximation (7), will be

$$\Delta G_{sr}(\omega) = -\frac{2dVc}{(4\pi)^2 R_1 R_2 c^3} \omega^2 e^{-ik(R_1 + R_2)}$$

$$= -\frac{2dV\Delta c}{(4\pi)^2 R_1 R_2 c^3} \omega^2 e^{-i\omega(T_1 + T_2)},$$

(A2)

where $T_1 = R_1/c$ and $T_2 = R_2/c$.

The background pressure at the receiver in the time domain is given by the inverse Fourier transform (8), which by substitution of (A1) becomes

$$p_s(t) = \frac{p_s(t-T)}{4\pi R} = \frac{\tilde{p}_s(t-T)e^{i\omega_0(t-T)}}{4\pi R},$$

(A3)

where $p_s(t)$ is the pressure at the source in the time domain, and $\omega_0$ is the central circular frequency of the source; an overtilde denotes demodulated quantities.

The perturbation of the pressure at the receiver in the time domain can be obtained by substituting the expression (A2) into Eq. (11),
Thus, the time–domain pressure anomaly due to the scatterer is proportional to the second time derivative of the background pressure at the receiver, centered about the time instant \( t_1 + t_2 \) instead of \( T \). The pressure \( p_s(t-T_1-T_2) \) can be written in terms of the demodulated source pressure,

\[
p_s(t-T_1-T_2) = \tilde{p}_s(t-T_1-T_2) e^{-i\omega_0(T_1+T_2)}.
\]

(A5)

Accordingly, its first and second derivatives can be written as follows:

\[
p_s'(t-T_1-T_2) = e^{i\omega_0(t-T_1-T_2)} [i \omega_0 \tilde{p}_s(t-T_1-T_2) + \tilde{p}_s'(t-T_1-T_2)],
\]

\[
p_s''(t-T_1-T_2) = e^{i\omega_0(t-T_1-T_2)} [-\omega_0^2 \tilde{p}_s(t-T_1-T_2) + i2 \omega_0 \tilde{p}_s'(t-T_1-T_2) + \tilde{p}_s''(t-T_1-T_2)].
\]

(A6)

(A7)

Thus, by combining Eqs. (A3) and (A4) and using demodulated quantities, the perturbed pressure at the receiver in the time domain can be written in the form

\[
p_s(t) + \Delta p_s(t) = \frac{p_s(t-T)}{4\pi R} + \frac{2dV\Delta c}{(4\pi)^2 R_1 R_2 c^3_2} p_s''(t-T_1-T_2)
\]

\[
= \frac{\tilde{p}_s(t-T) e^{i\omega_0(t-T)}}{4\pi R} + \frac{2dV\Delta c}{(4\pi)^2 R_1 R_2 c^3_2 e^{i\omega_0(t-T_1-T_2)}}
\]

\[
\times \left\{ \begin{array}{l}
-\omega_0^2 \tilde{p}_s(t-T_1-T_2) + i2 \omega_0 \tilde{p}_s'(t-T_1-T_2) + \tilde{p}_s''(t-T_1-T_2) \\
\end{array} \right\}.
\]

(A8)

The squared amplitude of the perturbed pressure, i.e., the squared arrival pattern, will be

\[
A(t) = a^2(t) = | p_s(t) + \Delta p_s(t) |^2
\]

\[
= \frac{2dV\Delta c}{(4\pi)^2 R_1 R_2 c^3_2} \cos(\omega_0\tau) 2\omega_0 \tilde{p}_s'(t-T_1-T_2)
\]

\[
- \frac{2dV\Delta c}{(4\pi)^2 R_1 R_2 c^3_2} \sin(\omega_0\tau) (\omega_0^2 \tilde{p}_s(t-T_1-T_2)
\]

\[
- \tilde{p}_s''(t-T_1-T_2) \right]^2.
\]

(A9)

where \( \tau = T_1 - T_2 \) is the differential travel time between the straight line from the source to the receiver and the line passing through the scatterer (\( \tau = 0 \)). By retaining first-order terms with respect to \( \Delta c \), the following expression can be obtained for \( A(t) \):

\[
A(t) = \frac{\tilde{p}_s^2(t-T)}{(4\pi R)^2} - \frac{2\tilde{p}_s(t-T)}{4\pi R} \frac{2dV\Delta c}{(4\pi)^2 R_1 R_2 c^3_2} \cos(\omega_0\tau)
\]

\[
\times [ \omega_0^2 \tilde{p}_s(t-T_1-T_2) - \tilde{p}_s''(t-T_1-T_2)]
\]

\[
- \frac{2\tilde{p}_s(t-T)}{4\pi R} \frac{2dV\Delta c}{(4\pi)^2 R_1 R_2 c^3_2} \sin(\omega_0\tau) 2\omega_0 \tilde{p}_s'(t-T_1-T_2).
\]

(A10)

The first term on the right-hand side is the unperturbed squared arrival pattern whereas the other two terms represent the perturbation of the squared arrival pattern due to the scatterer.

In the following, a number of cases is considered depending on the value of \( \omega_0\tau \), i.e., on the position of the scatterer, shedding light to the travel-time sensitivity behavior. Assuming \( \tilde{p}_s \) to be a real-valued pulse (a spike) with its maximum at zero, then \( \tilde{p}_s'' \) will have a negative maximum at zero whereas \( \tilde{p}_s' \) will be a decreasing antisymmetric function about zero.

(a) \( \tau = 0 \): In this case the scatterer lies on the straight line from the source to the receiver and the third term in (A10) vanishes due to the effect of \( \sin(\omega_0\tau) \). Since \( T = T_1 + T_2 \), the expression for \( A(t) \) becomes

\[
A(t) = \frac{\tilde{p}_s^2(t-T)}{(4\pi R)^2} - \frac{2\tilde{p}_s(t-T)}{4\pi R} \frac{2dV\Delta c}{(4\pi)^2 R_1 R_2 c^3_2} [\omega_0^2 \tilde{p}_s(t-T)
\]

\[
- \tilde{p}_s''(t-T)].
\]

(A11)

Since the functions \( \tilde{p}_s \) and \( \tilde{p}_s'' \) have their positive/negative maximum at zero, the perturbed squared arrival pattern in this case will have its maximum at \( t = T \). The bracketed term in the vicinity of \( t = T \) is positive. Accordingly, a sound-speed increase (\( \Delta c > 0 \)) will reduce the arrival amplitude whereas a sound-speed decrease (\( \Delta c < 0 \)) will cause amplification. The arrival time will remain unaffected (zero sensitivity).

(b) \( 0 < \omega_0\tau > -\pi/2 \): In this case the scatterer lies slightly off the straight line connecting the source and the receiver and both scattering terms in (A10) are active and centered at \( T_1 + T_2 > T \), i.e., at the right flank of the unperturbed arrival. In the case of a positive sound-speed perturbation (\( \Delta c > 0 \)) the second term in (A10)
will be negative whereas the factor of \( \hat{p}_s^c \) in the third term will be positive, since \( \sin(\omega_0 t) < 0 \). This means that the positive forms \( \hat{p}_s \) and \(-\hat{p}_s^c\) (centered at \( T_1 + T_2 > T \)) are subtracted from the unperturbed arrival (centered at \( T \)), whereas the decreasing from \( \hat{p}_s^c \) is added. This will cause the arrival maximum to move to the left, i.e., to smaller time values. In the case of a negative sound-speed perturbation (\( \Delta c < 0 \)) the opposite operations will cause a shift at the arrival maximum to larger arrival times (negative sensitivity).

(c) \( \omega_0 t = -\pi/2 \): In this case the scatterer lies further off the straight line connecting the source and the receiver, and now only the second scattering term in (A10) will be active. As in the previous case this case is characterized by negative sensitivity too.

(d) \(-\pi/2 > \omega_0 t > -\pi \): The term \( \cos(\omega_0 t) \) now attains negative values, starting from zero \( (\omega_0 t = -\pi/2) \) and going to \(-1 \) \( (\omega_0 t = -\pi) \). For \( \omega_0 t = -\pi/2 \) (the previous case) the second scattering term in (A10) dominates and gives negative sensitivity, whereas for \( \omega_0 t = -\pi \) it is the first scattering term that dominates, giving positive sensitivity, as described below. Thus, there should be a value for \( \omega_0 t \) in the open interval \( (-\pi, -\pi/2) \) where zero sensitivity is attained. This value will depend on the characteristics of the emitted signal \( \hat{p}_s \).

(e) \( \omega_0 t = -\pi \): In this case it is the first scattering term that dominates in (A10)—the second scattering term disappears since \( \sin(-\pi) = 0 \). The expression for \( A(t) \) becomes

\[
A(t) = \frac{\hat{p}_s^2(t-T)}{(4\pi R)^2} + \frac{2dV\Delta c}{(4\pi R)^2} R_2 c^3 \left[ \frac{\omega_0^2 \hat{p}_s(t-T) - \hat{p}_s^c(t-T)}{2} \right] \]  

(A12)

A positive sound-speed perturbation \( \Delta c > 0 \) will cause the positive forms \( \hat{p}_s \) and \(-\hat{p}_s^c\) centered at \( T_1 + T_2 > T \), i.e., at the right flank of the unperturbed arrival to be added to it, thus moving the arrival maximum to the right, i.e., to larger time values. A negative sound-speed perturbation \( \Delta c < 0 \) will cause the positive forms to be subtracted from the unperturbed arrival, thus moving the arrival maximum to the left, i.e., to smaller time values (positive sensitivity).

The condition \( \omega_0 t = -\pi \) defined the Fresnel radius \( R_F \), i.e., the radius of the first Fresnel zone.\(^{34,35}\) Thus, the outer boundary of the first Fresnel zone is a surface of positive travel-time sensitivity. The negative-sensitivity core has an outer radius larger than (but close to) \( R_F/\sqrt{2} \) and smaller than \( R_F \), the precise value depending on the characteristics of the emitted signal.

**APPENDIX B: VERTICAL SENSITIVITY OF PEAK ARRIVAL TIMES**

A wave-theoretic expression for the vertical sensitivity kernel of peak arrival times is derived in the following by assuming range independence for both the background environment \( c(z) \) and the sound-speed perturbation \( \Delta c(z) \). Since the medium preserves its horizontal stratification the Green’s

\[
\Delta k_n = -\frac{\rho \omega^2}{k_n} \int_0^h \Delta c(z') c^3(z') n^2(z') dz' 
\]

(B1)

and

\[
\Delta u_n(z) = -2\rho \omega^2 \int_0^h \Delta c(z') c^3(z') \frac{\sum_{m=1}^M U_{nm}(z') u_m(z) u_m(z)}{k_n k_m} dz'.
\]

(B2)

where \( h \) is the water depth. Taking a first-order perturbation of the normal-mode representation (19), with respect to eigenvalues and eigenfunctions, and substituting the relations (B1) and (B2), the perturbation of the Green’s function in the range independent case can be related to the underlying sound-speed perturbations as follows:

\[
\Delta G = -\int_0^h \sum_{m=1}^M \left[ \int_0^h \frac{U_{mn}(z') u_m(z')}{k_n} \right] \frac{1}{2} k_n + jR \right) \times U_{nm}(z') e^{-jk_nR} \frac{\omega^2}{2\pi R} c^3(z') dz',
\]

where \( h = u_n(z) + u_n(z), u_m(z) \) for \( n \neq m \) and \( u_m(z) \). By substituting the expression above into the Fourier transform (11), the corresponding perturbation of the pressure at the receiver in the time domain can be obtained. By combining this with the relation (14) the perturbation of the peak arrival time \( \Delta \tau_i \) can be written in the form

\[
\Delta \tau_i = \int_0^h S_i(z') \Delta c(z') dz'
\]

(B4)

where

\[
S_i(z') = \frac{1}{2} \int_{-\infty}^{\infty} \left[ j \omega (v_i - j\omega) + (\hat{v}_i - j\omega) \right] L(z';\omega) e^{i\omega t_i} d\omega
\]

and

\[
L(z';\omega) = \frac{\omega^2 P_i(\omega) e^{-j\pi/4}}{c^3(z') \sqrt{2\pi R}} \times \sum_{n=1}^M \left[ \int_0^h \frac{U_{nm}(z') u_m(z')}{k_n k_m} \right] \frac{\omega^2}{2\pi R} \frac{1}{2} k_n + jR \right) U_{nm}(z') e^{-jk_nR} \frac{\omega^2}{2\pi R} c^3(z') dz'.
\]

(B6)
The quantity $S_i(z')$ is the vertical sensitivity kernel of the peak arrival time $t_i$ with respect to depth. This is equivalent to integrating the 3-D travel-time sensitivity kernel $K$, over range and azimuth.

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22 P. M. Morse and H. Feshbach, Methods of Theoretical Physics (McGraw-Hill, New York, 1953).
32 The notation ±n is used as a ray identifier, where ± is the sign of grazing angle of the eigenray at the source (+ for upward, − for downward), and n is the number of turning points (upper and lower) of the eigenray.