Error estimates for finite element methods for the shallow water equations

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Abstract. We consider a simple initial–boundary–value problem for the system of the shallow water equations and its symmetric variant in one space dimension. We discretize the problem in space by the standard Galerkin–finite element method and prove error estimates for the resulting semidiscretizations for quasiuniform and uniform meshes. We also discretize the problem in time using the third–order Shu–Osher Runge–Kutta scheme and prove error estimates of optimal temporal order under a Courant stability condition.

1 Introduction

We consider the following initial–boundary–value problem for the one–dimensional shallow water equations: We seek $\eta = \eta(x,t)$ and $u = u(x,t)$, $0 \le x \le 1$, $0 \le t \le T$ such that

$$\begin{aligned}
&\eta_t + u_x + (\eta u)_x = 0 \,, \qquad 0 \le x \le 1 \,, \qquad 0 \le t \le T \,, \\
&u_t + \eta_x + u u_x = 0 \,, \qquad 0 \le x \le 1 \,, \qquad 0 \le t \le T \,, \\
&\eta(x,0) = \eta_0(x) \,, \quad u(x,0) = u_0(x) \,, \qquad 0 \le x \le 1 \,, \\
&u(0,t) = 0 \,, \quad u(1,t) = 0 \,, \qquad 0 \le t \le T \,.
\end{aligned} \tag{1}$$

Local in time existence and uniqueness of H^2 solutions for (1) was proved in [4] under the hypothesis that $1+\eta_0(x) > 0, x \in [0,1]$, which also guarantees that positivity of $1 + \eta$ is preserved during the life span of the solution of (1).

We also consider a *symmetric* variant of the system, given by

$$\begin{aligned}
&\eta_t + u_x + \tfrac{1}{2}(\eta u)_x = 0 \,, \qquad 0 \le x \le 1 \,, \qquad 0 \le t \le T \,, \\
&u_t + \eta_x + \tfrac{3}{2} u u_x + \tfrac{1}{2}\eta\eta_x = 0 \,, \qquad 0 \le x \le 1 \,, \qquad 0 \le t \le T \,, \\
&\eta(x,0) = \eta_0(x) \,, \quad u(x,0) = u_0(x) \,, \qquad 0 \le x \le 1 \,, \\
&u(0,t) = 0 \,, \quad u(1,t) = 0 \,, \qquad 0 \le t \le T \,,
\end{aligned} \tag{2}$$

for which local in time existence and uniqueness of H^2 solutions may be proved analogously cf. [1]. In [1] we derive this system and show that its solutions approximate those of (1) upon a nonlinear change of variable in the case of small amplititude scaling, motivated by the analogous results for dispersive equations of [2].

In the sequel, assuming that (1), (2) have sufficiently smooth solutions, we shall discretize these systems in the spatial variable using standard Galerkin–finite element methods and analyze the errors of the semidiscretization in the case of quasiuniform and uniform meshes. We shall also consider full discretizations of the systems by the third–order explicit Runge–Kutta scheme due to Shu and Osher [5] and analyze its errors. Due to space limitationa, here we shall only include statements of results; the interested reader may consult the proofs in [1].

2 Semidiscrete problems

Let $0 = x_1 < x_2 < \cdots < x_{N+1} = 1$ denote a quasiuniform partition of $[0, 1]$ with $h = \max_i(x_{i+1} - x_i)$, and for integers r, k such that $r \geq 2$, $0 \leq k \leq r - 2$, let $S_h = \{\phi \in C^k : \phi\big|_{[x_i,x_{i+1}]} \in P_{r-1},\ 1 \leq j \leq N\}$ and $S_{h,0} = \{\phi \in S_h,\ \phi(0) = \phi(1) = 0\}$. We discretize (1) by the standard Galerkin method, seeking $\eta_h : [0, T] \to S_h$, $u_h : [0, T] \to S_{h,0}$ such that for $t \in [0, T]$

$$\begin{aligned} &(\eta_{ht}, \phi) + (u_{hx}, \phi) + ((\eta_h u_h)_x, \phi) = 0\ ,\ \forall \phi \in S_h\ , \\ &(u_{ht}, \chi) + (\eta_{hx}, \chi) + (u_h u_{hx}, \chi) = 0\ ,\ \forall \chi \in S_h\ , \end{aligned} \tag{3}$$

where $(\cdot, \cdot)$ denotes the inner product on $L^2 = L^2(0, 1)$. We supplement (3) with the initial conditions

$$\eta_h(0) = P\eta_0\ , \qquad u_h(0) = P_0 u_0\ , \tag{4}$$

where P, P_0 are the L^2–projections onto S_h, $S_{h,0}$, respectively.

Analogously we discretize (2) seeking $(\eta_h(t), u_h(t)) \in S_h \times S_{h,0}$ for $0 \leq t \leq T$ such that

$$\begin{aligned} &(\eta_{ht}, \phi) + (u_{hx}, \phi) + \tfrac{1}{2}((\eta_h u_h)_x, \phi) = 0\ ,\ \forall \phi \in S_h\ , \\ &(u_{ht}, \chi) + (\eta_{hx}, \chi) + \tfrac{3}{2}(u_h u_{hx}, \chi) + \tfrac{1}{2}(\eta_h \eta_{hx}, \chi = 0\ ,\ \forall \chi \in S_h\ , \end{aligned} \tag{5}$$

with initial conditions given by (4) again.

It is well–known that for linear hyperbolic equations the standard Galerkin semidiscretization has an L^2 error estimate of $O(h^{r-1})$ that cannot be improved for general quasiuniform meshes. It is straightforward to prove an analogous result, cf. [1], for the *symmetric* system. (Here $\|\cdot\|$ denotes the L^2 norm.)

Proposition 2.1 Let (η, u) be the solution of (2). Then (5)–(4) has a unique solution (η_h, u_h) for $0 \leq t \leq T$ such that

$$\max_{0\leq t\leq T} (\|\eta(t) - \eta_h(t)\| + \|u(t) - u_h(t)\|) \leq Ch^{r-1}\ . \qquad \square$$

(Here and in the sequel C will denote generic constants independent of the discretization parameters.)

For the usual, nonsymmetric shallow–water system (1) the proof in [1] is more complicated, assumes that $r \geq 3$ (i.e. at least quadratics, cf. also [3]) and uses the positivity of $1 + \eta$:

Proposition 2.2 Let (η, u) be the solution of (1) satisfying $1+\eta > 0$ for $t \in [0,T]$, and suppose that $r \geq 3$ and that h is sufficiently small. Then (3)–(4) has a unique solution (η_h, u_h) for $0 \leq t \leq T$ satisfying

$$\max_{0\leq t\leq T} (\|\eta(t) - \eta_h(t)\| + \|u(t) - u_h(t)\|) \leq Ch^{r-1} . \qquad \square$$

Numerical experiments (cf. [1]) indicate that the assumption $r \geq 3$ may not be needed and that the error estimate for (1) holds for $r = 2$, i.e. for P_1 elements, as well.

In the case of *uniform mesh*, i.e. when $x_i = (i-1)h$, $1 \leq i \leq N+1$, $h = 1/N$, we may prove better estimates using appropriate superaccuracy (superconvergence) properties of the L^2 projections on S_h, $S_{h,0}$ of smooth functions satisfying appropriate boundary conditions. Let us consider the case $r = 2$, i.e. P_1 elements on a uniform mesh, and the system (1) whose solutions are assumed to be sufficiently smooth (at least C^4 and such that $1+\eta > 0$ for $t \in [0,T]$). We also assume that the initial data of (1) atisfies the compatibility conditions $\eta_0' \in C_0^3$, $u_0'' \in C_0^2$. (Here C_0^k are the C^k functions on $[0,1]$ whose values at $x = 0$ and at $x = 1$ are zero.) Then, cf. [1], the following result holds:

Proposition 2.3 Let (η, u) be the solution of (1) and suppose that the above–stated conditions hold. Then (η_h, u_h), the solution of (3)–(4) for a uniform mesh and $r = 2$, satisfies

$$\max_{0\leq t\leq T} (\|\eta(t) - \eta_h(t)\| + \|u(t) - u_h(t)\|) \leq Ch^2 . \qquad \square$$

An analogous result (under the compatibility conditions $\eta_0' \in C_0^3$, $\eta_0''' \in C_0^1$, $u_0'' \in C_0^2$) holds for (2) as well. Numerical experiments in [1] suggest that optimal–order error bounds in the case of uniform mesh hold for higher–order finite element spaces too, e.g. for cubic splines ($r = 4$).

Remark. In the case of *periodic* boundary conditions, it is straightforward to prove by Fourier techniques, cf. [1], that for both systems the L^2–errors are of $O(h^r)$, i.e. of optimal order, for $r \geq 2$.

3 Time stepping

We discretize in the temporal variable the o.d.e. systems represented by (3)–(4) or (5)–(4) by the explicit, third–order Runge–Kutta scheme due to Shu and Osher, [5]. The scheme has been used widely for time stepping in numerical methods for conservation laws with finite volume or discontinuous Galerkin spatial discretizations; in the case of the o.d.e. $y' = f(t,y)$ it takes the form

$$\begin{aligned} y^{n,1} &= y^n + kf(t^n, y^n) , \\ y^{n,2} &= y^n + \tfrac{k}{4} f(t^n, y^n) + \tfrac{k}{4} f(t^{n+1}, y^{n,1}) , \\ y^{n+1} &= y^n + \tfrac{k}{6} f(t^n, y^n) + \tfrac{k}{6} f(t^{n+1}, y^{n,1}) + \tfrac{2k}{3} f\left(t^n + \tfrac{k}{2}, y^{n,2}\right) , \end{aligned} \qquad (6)$$

where $t^n = nk$, $n = 0, 1, 2, \ldots$, k is the time step, and $y^n \approx y(t^n)$. For example, if $Mk = T$, the following is proved in [1].

Proposition 3.1 Let (η, u) be the solution of (2) and (H_h^n, U_h^n) be the fully discrete approximation of (2) resulting from the application of the time–stepping scheme (6) to the o.d.e. system (5)–(4) with $H_h^0 = P_{\eta^0}$, $U_h^0 = P_0 u^0$, in the case of a quasiuniform spatial mesh and for $r > 3$. If $\lambda = k/h$, there exist constants λ_0 and C, independent of k and h, such that for $\lambda \leq \lambda_0$,

$$\max_{0\leq n\leq M} \left(\|\eta(t^n) - H_h^n\| + \|u(t^n) - U_h^n\|\right) \leq C\left(k^3 + h^{r-1}\right) . \qquad \square$$

An analogous result holds for (1) as well.

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