

On the Construction and Properties of Weak Solutions Describing Dynamic Cavitation

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Abstract We consider the problem of dynamic cavity formation in isotropic compressible nonlinear elastic media. For the equations of radial elasticity we construct self-similar weak solutions that describe a cavity emanating from a state of uniform deformation. For dimensions $d = 2, 3$ we show that cavity formation is necessarily associated with a unique precursor shock. We also study the bifurcation diagram and do a detailed analysis of the singular asymptotics associated to cavity initiation as a function of the cavity speed of the self-similar profiles. We show that for stress free cavities the critical stretching associated with dynamically cavitating solutions coincides with the critical stretching in the bifurcation diagram of equilibrium elasticity. Our analysis treats both stress-free cavities and cavities with contents.

Keywords Cavitation · Shock wave · Polyconvex elasticity

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1 Introduction

The motion of a continuous medium with nonlinear elastic response is described by the system of partial differential equations

$$y_{tt} - \operatorname{div} \frac{\partial W}{\partial F}(\nabla y) = 0, \quad (1.1)$$

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where $y : \mathbb{R}^d \times \mathbb{R}_+ \rightarrow \mathbb{R}^d$ stands for the motion, $F = \nabla y$ is the deformation gradient, and we have employed the constitutive theory of hyperelasticity, $S = \frac{\partial W}{\partial F}(F)$, that the Piola-Kirchhoff stress S is given as the gradient of a stored energy function

$$W : M_+^{d \times d} := \{ F \in \mathbb{R}^{d \times d} : \det(F) > 0 \} \rightarrow \mathbb{R}.$$

For isotropic elastic materials the stored energy reads $W(F) = \Phi(v_1, v_2, \dots, v_d)$, where Φ is a symmetric function of the eigenvalues v_1, \dots, v_d of the positive square root $(F^T F)^{\frac{1}{2}}$; see [1, 22]. In that case (1.1) admits solutions that are radially symmetric motions,

$$y(x, t) = w(|x|, t) \frac{x}{|x|}, \quad R = |x|,$$

and are generated by solving for the amplitude $w : \mathbb{R}_+ \times \mathbb{R}_+ \rightarrow \mathbb{R}_+$ the scalar second-order equation

$$w_{tt} = \frac{1}{R^{d-1}} \frac{\partial}{\partial R} \left(R^{d-1} \frac{\partial \Phi}{\partial v_1} \left(w_R, \frac{w}{R}, \dots, \frac{w}{R} \right) \right) - \frac{1}{R} (d-1) \frac{\partial \Phi}{\partial v_2} \left(w_R, \frac{w}{R}, \dots, \frac{w}{R} \right). \tag{1.2}$$

This equation admits the special solution $w_h(R, t) = \lambda R$ corresponding to a homogeneous deformation of stretching $\lambda > 0$. The question was posed [2] if discontinuous solutions of (1.2) can be constructed and it has been tied to a possible explanation of the phenomenon of cavitation in stretched rubbers [6, 7].

Ball [2] in a seminal paper proposed to use continuum mechanics for modeling cavitation and used methods of the calculus of variations and bifurcation theory to construct cavitating solutions for the equilibrium version of (1.2): There is a critical stretching λ_{cr} such that for $\lambda < \lambda_{cr}$ the homogeneous deformation is the only minimizer of the elastic stored energy; by contrast, for $\lambda > \lambda_{cr}$ there exist nontrivial equilibria corresponding to a (stress-free) cavity in the material with energy less than the energy of the homogenous deformation [2]. We refer to [13, 14, 16, 17, 20, 21] and references therein for developments concerning equilibrium or quasistatic cavitating solutions.

In an important development, K.A. Pericak-Spector and S. Spector [18, 19] use the self-similar *ansatz*

$$w(R, t) = t\varphi\left(\frac{R}{t}\right) \tag{1.3}$$

to construct a weak solution for the dynamic problem (1.2) that corresponds to a spherical cavity emerging at time $t = 0$ from a homogeneously deformed state. The cavitating solution is constructed in dimension $d \geq 3$ for special classes of polyconvex energies [18, 19] and sufficiently large initial stretching. Remarkably, the cavitating solution has lower mechanical energy than the associated homogeneously deformed state from where it emerges [18], and thus provides a striking example of nonuniqueness of entropy weak solutions (for polyconvex energies). The dynamic cavitation problem is a little studied subject. Apart from [18, 19], there is an interesting almost explicit example of a dynamic solution that oscillates constructed by Chou-Wang and Horgan [4] for the dead load problem of an incompressible elastic material. Due to the incompressibility constraint the response is markedly different from the compressible case: beyond a critical load a cavity opens and then closes again, see [4]. The reader is referred to Choksi [3] for a discussion of the limit from compressible to incompressible response in radial elasticity, and to Hilgers [10] for other examples of non-uniqueness in multi-dimensional hyperbolic conservation laws due to radial point singularities.

The objective of the present work is to complement [18, 19] by establishing various further properties of weak solutions describing dynamic cavitation. First, we prove that cavity formation is always associated with a precursor shock, namely it is not possible to construct a cavitating solution that connects “smoothly” to a uniformly deformed state. Second, we study the bifurcation diagram for dynamically cavitating solution and provide a formula that determines the critical stretch required for opening a cavity. The critical stretch turns out (for traction free cavities) to be the same as that predicted from the equilibrium cavitation analysis of Ball [2]. In a companion paper [8] we reassess the issue of nonuniqueness of weak solutions, and show that local averaging of the cavitating weak solution contributes a surface energy when opening a cavity that renders the uniform deformation the energetically preferred solution, see [8] for details and comments on the ramifications.

We now provide an outline of the technical contents of the article: Throughout we work with stored energies of the form

$$\Phi(v_1, v_2, \dots, v_d) = \sum_{i=1}^d g(v_i) + h(v_1 v_2 \dots v_d), \tag{H0}$$

where $g(x) \in C^3[0, \infty)$, $h(x) \in C^3(0, \infty)$ satisfy

$$g''(x) > 0, \quad h''(x) > 0, \quad \lim_{x \rightarrow \infty} h(x) = +\infty, \tag{H1}$$

$$g'''(x) \leq 0, \quad h'''(x) < 0. \tag{H2}$$

Hypothesis (H1) refers to polyconvexity [2], while (H2) indicates elasticity with softening. The reader is referred to Appendix 7.2 where properties of isotropic stored energies are reviewed.

Following [18, 19], we introduce the self-similar *ansatz* (1.3) and the problem of cavity formation becomes to find solutions of the problem

$$\left(s^2 - \frac{\partial^2 \Phi}{\partial v_1^2}\right) \bar{\varphi} = \frac{d-1}{s} \left[\left(\dot{\varphi} - \frac{\varphi}{s}\right) \frac{\partial^2 \Phi}{\partial v_1 \partial v_2} + \frac{\partial \Phi}{\partial v_1} - \frac{\partial \Phi}{\partial v_2} \right], \tag{1.4}$$

$$\varphi_0 := \lim_{s \rightarrow 0^+} \varphi(s) > 0, \tag{1.5}$$

and to check whether such solutions can be connected to a uniformly deformed state, namely

$$\varphi(s) = \lambda s \quad \text{for } s > \sigma. \tag{1.6}$$

Here, $\varphi_0 > 0$ represents the speed of the cavity surface, $\lambda > 0$ the stretching of the (initially) uniform deformation and σ is the shock speed. We remark that (1.4) with (1.6) admit the special solution $\bar{\varphi} = \lambda s$ corresponding to a homogenous deformation $y_i(x) = \lambda x$; therefore, according to this scenario, cavity formation is associated to nonuniqueness for the initial value problem of the radial elasticity equation (1.2).

To make the problem (1.4)–(1.5) determinate it is necessary to specify the value of the radial component of the Cauchy stress $T_{rad}(0)$ at the cavity surface. Two types of boundary conditions are pursued (see Sect. 3.1) corresponding to stress-free cavities or to a cavity with content:

$$\begin{aligned} \text{either } T_{rad}(0) = 0 & \iff \lim_{s \rightarrow 0} \dot{\varphi} \left(\frac{\varphi}{s}\right)^{d-1} = h'^{-1}(0) := H \quad (\text{stress-free cavity}), \\ \text{or } T_{rad}(0) = G(\varphi_0) & \iff \lim_{s \rightarrow 0} \dot{\varphi} \left(\frac{\varphi}{s}\right)^{d-1} = h'^{-1}(G(\varphi_0)) \quad (\text{cavity with content}). \end{aligned} \tag{1.7}$$

Under the growth condition $|g'(x)| \leq C|x|^{d-2}$ (see (H3) in Sect. 3.1) and for dimension $d \geq 2$, the problem (1.4), (1.5) and (1.7) is desingularized at the origin and a solution $\varphi(s)$ is constructed (see Theorem 3.3). The question arises whether this cavitating solution can be connected to the uniform deformation (1.6) through a shock (or through a sonic singularity). This leads to studying the algebraic equation

$$\sigma = \sqrt{\left. \frac{\frac{\partial \Phi}{\partial v_1}(\dot{\varphi}, \frac{\varphi}{s}, \dots, \frac{\varphi}{s}) - \frac{\partial \Phi}{\partial v_1}(\frac{\varphi}{s}, \frac{\varphi}{s}, \dots, \frac{\varphi}{s})}{\dot{\varphi} - \frac{\varphi}{s}} \right|_{s=\sigma}}, \tag{1.8}$$

which manifests the Rankine-Hugoniot jump condition. In Theorems 3.4 and 4.1 we show there exists a unique σ where the connection can be effected, and that the connection either happens through a Lax shock or through a sonic singularity (i.e., a point where the coefficient $(s^2 - \Phi_{11})$ in (1.4) vanishes). Then, in Theorem 4.2, we restrict to dimensions $d = 2, 3$ and exclude the possibility of a connection through a sonic singularity. Our analysis is inspired and extends the results of [18, 19] where cavitating solutions are constructed for sufficiently large stretchings λ . In particular, we show that, for dimensions $d = 2, 3$, it is impossible to connect a cavitating solution smoothly to a uniformly deformed state and thus any cavitating solution is associated with a precursor shock.

The next objective is to study the bifurcation diagram of the cavitating weak solution and determine the critical stretching for dynamic cavitation. The bifurcation diagram is visualized as follows: The boundary condition (1.7) is expressed for the specific volume $v(s) := \det F = \dot{\varphi}(\frac{\varphi(s)}{s})^{d-1}$ in the general form $v(0) = V(\varphi_0)$. Given the cavity speed $\varphi_0 > 0$, let $(\varphi, v)(s; \varphi_0, V(\varphi_0))$ be the cavitating solution (constructed in Sect. 3.1) emanating from data $\varphi_0, v_0 = V(\varphi_0)$. Denoting by $\sigma = \sigma(\varphi_0, V(\varphi_0))$ the connection point, the associated stretching defines the map

$$\varphi_0 \mapsto \Lambda(\varphi_0, V(\varphi_0)) \quad \text{where} \quad \Lambda(\varphi_0, V(\varphi_0)) := \frac{\varphi(\sigma; \varphi_0, V(\varphi_0))}{\sigma}, \tag{1.9}$$

which is precisely the dynamic bifurcation diagram (see Fig. 2 for a numerical computation of this map). The limit $\lim_{\varphi_0 \rightarrow 0+} \Lambda(\varphi_0, V(\varphi_0))$ will determine the critical stretching. The technique of recovering the bifurcation point λ_{cr} by computing the cavitating solution and sending the inner radius of the cavity to zero is espoused in [15], where the authors use it to devise a numerical scheme for computing λ_{cr} in equilibrium elasticity.

To understand the limiting behavior of cavitating solutions as $\varphi_0 \rightarrow 0$, we introduce the rescaling

$$\psi(\xi; \varphi_0, V(\varphi_0)) := \frac{\varphi(\varphi_0 \xi; \varphi_0, V(\varphi_0))}{\varphi_0}, \quad \delta(\xi; \varphi_0, V(\varphi_0)) := v(\varphi_0 \xi; \varphi_0, V(\varphi_0)), \tag{1.10}$$

which captures the inner asymptotics of the cavitating solution (φ, v) to (1.4)–(1.7). Rescalings have been useful in the study of cavitation for equilibrium elasticity [2] and will play an instrumental role in determining the critical stretching for dynamic cavitation.

It is proved in Proposition 5.6 that the rescaled solutions converge to a limiting profile,

$$(\psi, \delta)(\xi; \varphi_0, V(\varphi_0)) \rightarrow (\psi_0, \delta_0)(\xi; V(0)), \quad \text{as } \varphi_0 \rightarrow 0_+,$$

uniformly on compact subsets of $[0, \infty)$.

The limiting profile $(\psi_0(\xi), \delta_0(\xi))$, where $\delta_0 = \psi_0'(\frac{\psi_0}{\xi})^{d-1}$, is defined on $[0, \infty)$ and solves the initial value problem

$$\begin{aligned}
 -\frac{\partial^2 \Phi}{\partial v_1^2} \left(\psi_0', \frac{\psi_0}{\xi}, \dots, \frac{\psi_0}{\xi} \right) \psi_0'' &= \frac{d-1}{\xi} \left(\psi_0' - \frac{\psi_0}{\xi} \right) \left[\frac{\partial^2 \Phi}{\partial v_1 \partial v_2} + \frac{\frac{\partial \Phi}{\partial v_1} - \frac{\partial \Phi}{\partial v_2}}{\psi_0' - \frac{\psi_0}{\xi}} \right] \\
 &\times \left(\psi_0', \frac{\psi_0}{\xi}, \dots, \frac{\psi_0}{\xi} \right), \tag{1.11} \\
 \psi_0(0) &= 1, \\
 \delta_0(0) &= V(0).
 \end{aligned}$$

The solvability of (1.11) and properties of its solutions are discussed in Proposition 5.5, where it is in particular shown that the (inner) solution is associated with a critical stretching at infinity

$$\Lambda_0(V(0)) := \lim_{\xi \rightarrow \infty} \frac{\psi_0(\xi; V(0))}{\xi}. \tag{1.12}$$

Equation (1.11)₁ is precisely the equation describing cavitating solutions in equilibrium radial elasticity, suggesting that the critical stretch for dynamic cavitation and equilibrium cavitation might conceivably coincide. The critical stretch λ_{cr} for cavitation in equilibrium radial elasticity is studied in [2, Sect. 7.5] where various representation formulas for λ_{cr} are established. In Sect. 5.3.2 we pursue this analogy, we show that for a stress-free cavity $\Lambda_0(H) = \lambda_{cr}$, and establish representation formulas for the critical stretch and corresponding lower bounds.

Finally, in Theorem 5.7, we study the behavior of the cavitating solution $\varphi(\cdot; \varphi_0, V(\varphi_0))$ and the associated stretch (1.9) as the cavity speed $\varphi_0 \rightarrow 0$. We establish that

$$\lim_{\varphi_0 \rightarrow 0_+} \Lambda(\varphi_0, V(\varphi_0)) = \Lambda_0(V(0)),$$

where $\Lambda_0(V(0))$ is given by (1.12), that the speed and the strength of the precursor shock satisfy

$$\begin{aligned}
 \lim_{\varphi_0 \rightarrow 0_+} \sigma(\varphi_0, V(\varphi_0)) &= \sqrt{\frac{\partial^2 \Phi}{\partial v_1^2}(\Lambda_0(V(0)), \dots, \Lambda_0(V(0)))}, \\
 \lim_{\varphi_0 \rightarrow 0_+} \left[\frac{\varphi}{s} - \dot{\varphi} \right] (\sigma(\varphi_0, V(\varphi_0))) &= 0,
 \end{aligned}$$

and that

$$\varphi(s; \varphi_0, V(\varphi_0)) \rightarrow \Lambda_0(V(0))s, \quad \text{as } \varphi_0 \rightarrow 0_+.$$

Our analysis proves that the critical stretching for equilibrium and dynamic cavitation coincide.

The structure of the article is as follows: In Sect. 2 we introduce the equations of radial elasticity for isotropic elastic materials. In Sect. 3 we derive the equations for self-similar solutions of radial elasticity, describe various special solutions, and present the problem of cavitation. The analysis of Sect. 3 follows the ideas and extends the analysis of [18, 19] to a more general class of (polyconvex) stored energies and to boundary conditions of cavities with content. Sections 4 and 5 contain the main new results. In Sect. 4 we establish various

properties of weak solutions describing cavity formation from a homogeneously deformed state. In Sect. 5 we study the bifurcation curves associated with cavitating weak solutions and establish the properties of the critical stretching and its relation to the critical stretching predicted by the equilibrium elasticity equation. The Appendix lists some properties of radial deformations, and collects information on stored energies that is widely used in various places of the text.

2 The Equations of Radial Elasticity

The stored energy of an isotropic elastic material has to satisfy the symmetry requirements

$$\begin{aligned} \text{frame indifference: } & W(QF) = W(F), \quad \forall Q \in SO(d), \\ \text{isotropy: } & W(F) = W(FQ), \quad \forall Q \in SO(d), \end{aligned}$$

where Q is any proper rotation. These requirements are equivalent to

$$W(F) = \Phi(v_1, v_2, \dots, v_d),$$

where $\Phi(v_1, v_2, \dots, v_d) : \mathbb{R}_{++}^d \rightarrow \mathbb{R}$ is a symmetric function of its arguments and v_1, \dots, v_d are the eigenvalues of $(F^T F)^{\frac{1}{2}}$ called principal stretches [1, 2, 22].

For isotropic materials the system of elasticity (1.1) admits radial solutions of the form

$$y(x, t) = w(R, t) \frac{x}{R} \quad \text{with } R = |x|. \tag{2.1}$$

The deformation gradient is computed by

$$\nabla y = w_R \frac{x \otimes x}{R^2} + \frac{w}{R} \left(\mathbf{I} - \frac{x \otimes x}{R^2} \right) \tag{2.2}$$

and has principal stretches $v_1 = w_R, v_2 = \dots = v_d = \frac{w}{R}$. Using results on spectral representations of functions of matrices one computes the first Piola-Kirchhoff stress [2, p. 564],

$$S(\nabla y) = \Phi_1 \left(w_R, \frac{w}{R}, \dots, \frac{w}{R} \right) \frac{x \otimes x}{R^2} + \Phi_2 \left(w_R, \frac{w}{R}, \dots, \frac{w}{R} \right) \left(\mathbf{I} - \frac{x \otimes x}{R^2} \right), \tag{2.3}$$

where we used the notation $\Phi_1 \equiv \frac{\partial \Phi}{\partial v_1}, \Phi_2 \equiv \frac{\partial \Phi}{\partial v_2}$ and the symmetry property (7.8). (The reader is referred to Appendix 7.2 for properties of the stored energies Φ and details on the notation used throughout.) Using the above formulas one computes that the amplitude w of the radial motion (2.1) is generated by solving the second-order partial differential equation

$$w_{tt} = \frac{1}{R^{d-1}} \frac{\partial}{\partial R} \left(R^{d-1} \frac{\partial \Phi}{\partial v_1} \left(w_R, \frac{w}{R}, \dots, \frac{w}{R} \right) \right) - \frac{d-1}{R} \frac{\partial \Phi}{\partial v_2} \left(w_R, \frac{w}{R}, \dots, \frac{w}{R} \right). \tag{2.4}$$

In order for solutions to be interpreted as elastic motions one needs to impose the requirement

$$\det F = w_R \left(\frac{w}{R} \right)^{d-1} > 0$$

on solutions of (2.4), which for radial motions suffices to exclude interpenetration of matter.

Equation (2.4) can also be derived by considering the action functional for radial, isotropic elastic materials, defined as the difference between kinetic and potential energy

$$I[w] := \int_0^T \int_0^1 R^{d-1} \left(\frac{1}{2} w_t^2 - \Phi \left(w_R, \frac{w}{R}, \dots, \frac{w}{R} \right) \right) dR dt.$$

Critical points of the functional $I[w]$ are obtained by computing the first variation and setting it to zero,

$$\left. \frac{d}{d\delta} \right|_{\delta=0} I[w + \delta\psi] = 0,$$

which gives the weak form of (2.4),

$$\int_0^T \int_0^1 R^{d-1} \left(w_t \psi_t - \Phi_1 \left(w_R, \frac{w}{R}, \dots, \frac{w}{R} \right) \psi_R - \frac{d-1}{R} \Phi_2 \left(w_R, \frac{w}{R}, \dots, \frac{w}{R} \right) \psi \right) dR dt = 0.$$

Finally, (2.4) can be expressed as a first order system by introducing the variables

$$a = w_R, \quad b = \frac{w}{R}, \quad v = w_t, \tag{2.5}$$

where u is the (longitudinal) strain, b is the transverse strain, and v is the velocity in the radial direction. It is expressed as the equivalent first order system

$$\begin{aligned} a_t &= v_R, \\ v_t &= \partial_R (\Phi_1(a, b, \dots, b)) + \frac{d-1}{R} (\Phi_1(a, b, \dots, b) - \Phi_2(a, b, \dots, b)), \\ b_t &= \frac{v}{R}, \end{aligned} \tag{2.6}$$

subject to the involution $(bR)_R = a$. This is a system of balance laws with geometric singularity at $R = 0$. Under the hypothesis $\Phi_{11} > 0$ the system (2.6) is hyperbolic (see [5, Definition 3.1.1] for the usual definition). The characteristic speeds $\lambda_{\pm} = \pm \sqrt{\Phi_{11}}$ are genuinely nonlinear, while $\lambda_0 = 0$ is linearly degenerate, see [5, 12]. The eigenvalues and the corresponding right and left eigenvectors of the flux of the system (2.6) are given by

$$\begin{aligned} \lambda_+ &= \sqrt{\Phi_{11}}, & r_+ &= (1, \sqrt{\Phi_{11}}, 0)^\top, & l_+ &= (\sqrt{\Phi_{11}}, 1, (d-1)\Phi_{12}/\sqrt{\Phi_{11}}), \\ \lambda_- &= -\sqrt{\Phi_{11}}, & r_- &= (1, -\sqrt{\Phi_{11}}, 0)^\top, & l_- &= (\sqrt{\Phi_{11}}, -1, (d-1)\Phi_{12}/\sqrt{\Phi_{11}}), \\ \lambda_0 &= 0, & r_0 &= ((d-1)\Phi_{12}, 0, -\Phi_{11})^\top, & l_0 &= (0, 0, 1). \end{aligned}$$

3 The Cavitating Solution of Pericak-Spector and Spector

We are interested in (2.4) subject to the initial-boundary conditions

$$\begin{cases} w(R, 0) = \lambda R, \\ w(R, t) = \lambda R, \quad \text{for } |R| > \bar{\sigma} t. \end{cases} \tag{3.1}$$

The symmetry of Φ implies that $\Phi_1(\lambda, \lambda) = \Phi_2(\lambda, \lambda)$ and the homogeneous deformation $w_h(R, t) = \lambda R$ is a special equilibrium solution of (2.4) associated to the stretching $\lambda > 0$.

To obtain additional solutions, it was suggested in [18] to exploit the invariance of (2.4), (3.1) under the family of the scaling transformations $w_\lambda(R, t) = \lambda w(\lambda R, \lambda t)$ and to seek solutions in self-similar form

$$w(R, t) = t\varphi\left(\frac{R}{t}\right). \tag{3.2}$$

Introducing the ansatz (3.2) to (2.4), and using the notations $s = \frac{R}{t}$ and $\dot{} = \frac{d}{ds}$, we may see that $\varphi(s)$ satisfies the singular second-order ordinary differential equation

$$(s^2 - \Phi_{11})\ddot{\varphi} = \frac{d-1}{s}\left(\dot{\varphi} - \frac{\varphi}{s}\right)\left[\Phi_{12} + \frac{\Phi_1 - \Phi_2}{\dot{\varphi} - \frac{\varphi}{s}}\right]. \tag{3.3}$$

Henceforth, we will be using the short hand notations

$$\Phi_i\left(\dot{\varphi}, \frac{\varphi}{s}\right) \equiv \frac{\partial \Phi}{\partial v_i}\left(\dot{\varphi}, \frac{\varphi}{s}, \dots, \frac{\varphi}{s}\right), \quad \Phi_{ij}\left(\dot{\varphi}, \frac{\varphi}{s}\right) \equiv \frac{\partial^2 \Phi}{\partial v_i \partial v_j}\left(\dot{\varphi}, \frac{\varphi}{s}, \dots, \frac{\varphi}{s}\right),$$

and so on for higher derivatives. We refer to Appendix 7.2 for details, and caution the reader that the notation together with the symmetry properties (7.8), (7.9) has implications on the differentiation of such formulas.

Moreover, we introduce the variables a, b defined in analogy to (2.5) by

$$a = \dot{\varphi}, \quad b = \frac{\varphi}{s},$$

and rewrite (3.3) in the form of the first order system

$$\begin{cases} (s^2 - \Phi_{11}(a, b))\dot{a} = \frac{(d-1)}{s}(a-b)P(a, b), \\ \dot{b} = \frac{1}{s}(a-b), \end{cases} \tag{3.4}$$

where

$$P(a, b) = \begin{cases} \Phi_{12}(a, b) + \frac{\Phi_1(a,b) - \Phi_2(a,b)}{a-b}, & a < b, \\ \Phi_{11}(b, b), & a = b, \end{cases} \tag{3.5}$$

is a continuous function on $\{(a, b) \in \mathbb{R}^2 : 0 < a \leq b\}$.

In analogy to the standard theory [12] of the Riemann problem for conservation laws, it is instructive to classify elementary solutions of (3.4). There are three classes of special solutions:

- (a) *Uniformly deformed states.* A special class of solutions of (3.4) are the constant states $a = b = \text{constant}$, which yield a uniform deformation $w_h(R) = \lambda R$ for the original system.
- (b) *Continuous solutions.* The balance of the convective and the production terms in (3.4) leads to a class of solutions that are continuous (which are not present in homogeneous conservation laws and are of different origin than the rarefaction waves). These will be the main object of study here. There are two features of (3.4) that need to be addressed by the analysis: (i) the geometric singularity at $s = 0$, and (ii) the difficulty emerging from a potential free boundary at the sonic curve $s = \pm\sqrt{\Phi_{11}(a, b)}$. It is well known that the resolution of the Riemann problem for multi-dimensional hyperbolic systems

leads to systems that change type across sonic-curves in the self-similar variables. The analog of this phenomenon for radial solutions leads to singular ordinary differential equations across the sonic lines.

(c) *Shocks*. One may express the system (3.4) in the equivalent form

$$\begin{aligned} \frac{d}{ds}(s^2 a - \Phi_1(a, b)) &= 2sa + \frac{d-1}{s}(\Phi_1(a, b) - \Phi_2(a, b)), \\ \frac{db}{ds} &= \frac{1}{s}(a - b). \end{aligned} \tag{3.6}$$

Two smooth branches of solutions to (3.6) might be connected through a jump discontinuity at $s = \sigma$ provided that the Rankine-Hugoniot jump conditions

$$b_- = b_+ =: b, \quad \sigma^2 = \frac{\Phi_1(a_+, b) - \Phi_2(a_-, b)}{a_+ - a_-} \tag{3.7}$$

are satisfied, where

$$(a_-, b_-) = \lim_{s \rightarrow \sigma_-} (a, b)(s), \quad (a_+, b_+) = \lim_{s \rightarrow \sigma_+} (a, b)(s).$$

According to the Lax shock admissibility criterion (see [12] where the criterion was introduced or [5, Sects. 8.3, 9.4]), a shock of the 2nd characteristic family will be admissible if

$$\sqrt{\Phi_{11}(a_+, b)} < \sigma_+ = \sqrt{\frac{\Phi_1(a_+, b) - \Phi_2(a_-, b)}{a_+ - a_-}} < \sqrt{\Phi_{11}(a_-, b)}. \tag{3.8}$$

In particular,

- if $\Phi_{111} > 0$, then (3.8) is equivalent to $a_+ < a_-$,
- if $\Phi_{111} < 0$, then (3.8) is equivalent to $a_+ > a_-$.

Similarly, shocks of the 1st characteristic family are admissible via the Lax criterion if

$$-\sqrt{\Phi_{11}(a_+, b)} < \sigma_- = -\sqrt{\frac{\Phi_1(a_+, b) - \Phi_2(a_-, b)}{a_+ - a_-}} < -\sqrt{\Phi_{11}(a_-, b)}, \tag{3.9}$$

in which case

- if $\Phi_{111} > 0$, then (3.9) is equivalent to $a_- < a_+$,
- if $\Phi_{111} < 0$, then (3.9) is equivalent to $a_- > a_+$.

For radial motions shocks of the 2nd characteristic family are outgoing while shocks of the 1st characteristic family are incoming to the origin. For the cavitation problem, it is natural to restrict to outgoing shocks and the kinematics of the cavity dictates that $a_- < a_+$. Therefore, we impose the condition $\Phi_{111} < 0$ which corresponds to softening elastic response. Softening refers to the property that the elastic modulus decreases with an increase of the longitudinal strain and plays an important role in cavitation analysis.

3.1 The Cavitating Solution

We next consider the problem of cavitation and discuss the continuous type of solutions in this context. We employ a constitutive relation of polyconvex class

$$\Phi(v_1, v_2, \dots, v_d) = \sum_{i=1}^d g(v_i) + h(v_1 v_2 \dots v_d), \tag{H0}$$

where $g \in C^3[0, \infty)$, $h \in C^3(0, \infty)$ satisfy (H1) and (H2) and thus $\Phi_{11} > 0$ and $\Phi_{111} < 0$. Hypothesis (H1) alludes to polyconvexity of the stored energy while Hypothesis (H2) manifests softening elastic response.

A stored energy of the form (H0) with $g(x) = \frac{1}{2}x^2$ was used in [18] to establish cavitation for $d \geq 3$. The generalization presented in (H0) is necessary in order to handle the case of $d = 2$, as the hypothesis of quadratic growth is too strong to allow for a cavity when $d = 2$. The ideas presented in this section closely follow the discussion of [18, 19], nevertheless they are presented here first for the reader’s convenience but also to set up the landscape for the forthcoming analysis in the following sections.

The differential equation (3.3) is expressed as

$$Q\left(\dot{\varphi}, \frac{\varphi}{s}, s\right)\ddot{\varphi} = \frac{(d-1)}{s}\left(\dot{\varphi} - \frac{\varphi}{s}\right)P\left(\dot{\varphi}, \frac{\varphi}{s}\right), \tag{3.10}$$

or equivalently

$$\begin{cases} Q(a, b, s)\dot{a} = \frac{(d-1)}{s}(a-b)P(a, b), \\ \dot{b} = \frac{1}{s}(a-b), \end{cases} \tag{3.11}$$

where

$$Q(a, b, s) = s^2 - \Phi_{11}(a, b) \stackrel{(H0)}{=} s^2 - [g''(a) + b^{2d-2}h''(ab^{d-1})]. \tag{3.12}$$

Desingularization at the Origin We next transform (3.10) into a system for the quantities

$$\varphi(s), v(s) = \dot{\varphi}\left(\frac{\varphi}{s}\right)^{d-1} \quad \text{with data } \varphi(0) = \varphi_0 > 0, v(0) = v_0 > 0, \tag{3.13}$$

henceforth restricting to stored energies of class (H0). A lengthy but straightforward calculation shows that (φ, v) satisfies the initial-value problem

$$\begin{cases} \dot{\varphi} = v\left(\frac{s}{\varphi}\right)^{d-1}, \\ \dot{v} = \left(\frac{d-1}{\varphi}\right)\frac{\left(\frac{s}{\varphi}\right)^{2d-3}v\left(v\left(\frac{s}{\varphi}\right)^d - 1\right)[s^2 - g''\left(v\left(\frac{s}{\varphi}\right)^{d-1}\right)]}{\{-h''(v) + [s^2 - g''\left(v\left(\frac{s}{\varphi}\right)^{d-1}\right)]\left(\frac{s}{\varphi}\right)^{2d-2}\}} \\ \quad + \left(\frac{d-1}{\varphi}\right)\frac{\left(\frac{s}{\varphi}\right)^{d-2}[g'\left(v\left(\frac{s}{\varphi}\right)^{d-1}\right) - g'\left(\frac{\varphi}{s}\right)]}{\{-h''(v) + [s^2 - g''\left(v\left(\frac{s}{\varphi}\right)^{d-1}\right)]\left(\frac{s}{\varphi}\right)^{2d-2}\}}, \\ \varphi(0) = \varphi_0 > 0, \\ v(0) = v_0 > 0. \end{cases} \tag{3.14}$$

In view of (H1) and the assumption $d \geq 2$, the only term on the right-hand side of (3.14) that might be singular at $s = 0$ is the term $g'(\frac{\varphi}{s})(\frac{s}{\varphi})^{d-2}$. This motivates to impose the growth condition

$$\lim_{x \rightarrow \infty} \left(\frac{g'(x)}{x^{d-2}} \right) = \gamma \geq 0. \tag{H3}$$

Doing that the emerging system is not singular and one may apply the standard existence theory for ordinary differential equations to obtain

Lemma 3.1 *Suppose $d \geq 2$ and hypotheses (H0), (H1), and (H3) hold. Then there exists a unique solution of the system (3.14) defined on a maximal interval of existence.*

Proof Under hypotheses (H0)–(H1) we have $g''(0) > 0, h''(v_0) > 0$. Moreover, by (H3), the limit $\lim_{x \rightarrow \infty} (g'(x)x^{2-d})$ exists and is finite and the right hand side of (3.14) is continuous for $s \in [0, 1]$ (up to the boundary $s = 0$). A careful review of the various terms indicates that the initial value problem (3.14) is expressed as

$$\begin{cases} \dot{\varphi} = A(s, v, \varphi), \\ \dot{v} = B(s, v, \varphi) + C(s, v, \varphi)H(s, \varphi), \\ \varphi(0) = \varphi_0 > 0, \\ v(0) = v_0 > 0, \end{cases} \tag{3.15}$$

where $A, B, C : [0, 1] \times \mathcal{N}(v_0, \varphi_0) \rightarrow \mathbb{R}$ are C^1 functions on $[0, 1] \times \mathcal{N}(v_0, \varphi_0)$ with \mathcal{N} a neighbourhood of (v_0, φ_0) . The term

$$H(s, \varphi) = \begin{cases} (\frac{s}{\varphi})^{d-2} g'(\frac{\varphi}{s}), & s > 0, \\ \gamma, & s = 0 \end{cases} \tag{3.16}$$

carries the singular behaviour in s and by (H3) it is continuous on $[0, 1] \times \mathcal{N}(v_0, \varphi_0)$.

Moreover, the assumptions $g'' > 0, g''' \leq 0$ imply

$$0 \leq g''(x)x \leq \int_0^1 g'(sx)x ds = g'(x) - g'(0), \quad x > 0$$

and, using once again (H3), it automatically implies

$$0 \leq \limsup_{x \rightarrow \infty} \left(\frac{g''(x)}{x^{d-3}} \right) \leq \gamma_0 = \begin{cases} \gamma - g'(0), & d = 2, \\ \gamma, & d \geq 3. \end{cases} \tag{3.17}$$

Now observe that

$$\left| \frac{\partial H}{\partial \varphi}(s, \varphi) \right| = \frac{1}{\varphi} \left| g''\left(\frac{\varphi}{s}\right) \left(\frac{s}{\varphi}\right)^{d-3} - (d-2)g'\left(\frac{\varphi}{s}\right) \left(\frac{s}{\varphi}\right)^{d-2} \right| \leq C$$

for $s \in [0, 1]$ and $\varphi \in \mathcal{N}(\varphi_0)$ a suitable neighbourhood of $\varphi_0 > 0$. The standard existence and uniqueness theory for systems of ordinary differential equations then provides the result. \square

Remark 3.2 The sign requirements $g'' > 0, g''' \leq 0$ in hypotheses (H1), (H2) place the restriction that $\limsup_{x \rightarrow \infty} |\frac{g'(x)}{x}| < \infty$. Also, if $d = 2$ then (H3) enforces that $\lim_{x \rightarrow \infty} |g'(x)| = \gamma < \infty$. This is consistent with (H1)–(H2).

Boundary Data at the Cavity Surface A natural assumption motivated from mechanical considerations is to impose that the radial Cauchy stress vanishes at the cavity surface. Using the standard formula relating the Cauchy stress tensor T to the Piola-Kirchhoff stress S (e.g., [1, 2])

$$T(F) = \frac{1}{\det F} S(F) F^\top$$

and (2.3) it follows that for ∇y given by (2.2) we have

$$T(\nabla y) = \frac{1}{w_R \left(\frac{w}{R}\right)^{d-1}} \left[\Phi_1 w_R \frac{x \otimes x}{R^2} + \Phi_2 \frac{w}{R} \left(\mathbf{I} - \frac{x \otimes x}{R^2} \right) \right].$$

The radial component of the Cauchy stress is given by

$$T_{rad}(s) := \frac{x}{R} \cdot T(\nabla y) \frac{x}{R} \stackrel{(H0)}{=} \left(\frac{s}{\varphi}\right)^{d-1} \left[g'(\dot{\varphi}(s)) + \left(\frac{\varphi}{s}\right)^{d-1} h'(v(s)) \right].$$

For the solution φ, v of (3.14) it is easy to see that

$$\frac{\varphi}{s} \sim \frac{\varphi_0}{s}, \quad \dot{\varphi}(s) \sim v_0 \left(\frac{s}{\varphi_0}\right)^{d-1} \quad \text{as } s \rightarrow 0_+$$

and therefore

$$\lim_{s \rightarrow 0_+} T_{rad}(s) = h'(v_0). \tag{3.18}$$

This motivates to impose the growth condition

$$h'(x) \rightarrow -\infty \quad \text{as } x \rightarrow 0_+, \quad h'(x) \rightarrow +\infty \quad \text{as } x \rightarrow +\infty. \tag{H4}$$

Under (H1), (H4) the inverse h'^{-1} is a well-defined function on \mathbb{R} and the boundary condition becomes

$$T_{rad}(0) = 0 \quad \text{is equivalent to} \quad v_0 = H := h'^{-1}(0). \tag{3.19}$$

One may consider more general boundary conditions that are referred in [18] as cavities with content and require that $T_{rad}(0) = G(\varphi_0)$, where G is some prescribed function. Such conditions could model at a phenomenological level the effect of remnant plasticity inside the cavity, and are postulated in analogy to the form of kinetic relations in the motion of phase boundaries. For physical reasons the remnant plasticity at the cavity should correspond to tensile forces, which dictates that $G(\varphi_0) > 0$. One checks that

$$T_{rad}(0) = G(\varphi_0) \quad \text{is equivalent to} \quad v_0 = h'^{-1}(G(\varphi_0)). \tag{3.20}$$

It is not entirely clear if such an assumption is mechanically justified, nevertheless it can be analyzed by the mathematical theory at no additional effort. Note that both (3.19) and (3.20) decrease the freedom of the data by one degree. For the bifurcation analysis in Sect. 5 we assume that $G(\varphi_0)$ is continuous at $\varphi_0 = 0$. This implies that $V(x) := h'^{-1}(G(x))$ is also continuous at $x = 0$.

A Class of C^2 Self-Similar Solutions We now construct a class of C^2 self-similar solutions to (3.10). Proceeding along the lines of [18, Theorem 5.1] we have:

Theorem 3.3 *Assume that $d \geq 2$, Φ satisfies (H0)–(H3) and let $\varphi_0 > 0, v_0 > 0$. Then, there exists a unique solution φ of (3.10) satisfying the initial data (3.13) and defined on a maximal interval of existence $[0, T)$, with $T < \infty$. The solution has the following properties:*

(i) (φ, v) solves (3.14) and there holds

$$\frac{\varphi(s)}{s} \sim \frac{\varphi_0}{s} \quad \text{and} \quad \dot{\varphi}(s) \sim v_0 \left(\frac{s}{\varphi_0}\right)^{d-1} \quad \text{as } s \rightarrow 0_+. \tag{3.21}$$

(ii) $\dot{\varphi}, \frac{\varphi}{s}, (\dot{\varphi} - \frac{\varphi}{s})$ are strictly monotonic and satisfy

$$\ddot{\varphi}(s) > 0, \quad \frac{d}{ds} \left(\frac{\varphi}{s}\right) < 0, \quad \frac{d}{ds} \left(\dot{\varphi} - \frac{\varphi}{s}\right) > 0 \quad \text{on } 0 < s < T, \tag{3.22}$$

$$0 < \dot{\varphi}(s) < \dot{\varphi}(t) < \frac{\varphi(t)}{t} < \frac{\varphi(s)}{s} \quad \text{for } 0 < s < t < T, \tag{3.23}$$

and

$$Q\left(\dot{\varphi}, \frac{\varphi}{s}, s\right) < 0, \quad \dot{\varphi}(s) - \frac{\varphi(s)}{s} < 0 \quad \text{for } 0 < s < T. \tag{3.24}$$

(iii) The following limits exist

$$\lim_{s \rightarrow T_-} Q\left(\dot{\varphi}, \frac{\varphi}{s}, s\right) = 0, \quad \lim_{s \rightarrow T_-} \left(\dot{\varphi} - \frac{\varphi}{s}\right) = c_0 \leq 0 \tag{3.25}$$

(but it is not known if $c_0 < 0$).

Proof By Lemma 3.1 there exists a unique local solution (φ, v) of (3.14) with $\varphi(0) = \varphi_0 > 0, v(0) = v_0 > 0$ which of course satisfies (i). Next, we observe that $(a, b) = (\dot{\varphi}, \frac{\varphi}{s})$ solves the system (3.11) where, in view of (3.12) and (3.21), we have $Q(0) < 0$. This together with (3.21) implies that $Q(s) < 0$ and $0 < a(s) < b(s)$ for $s \in (0, \varepsilon]$ for sufficiently small $\varepsilon > 0$. We now check that Φ satisfying (H0)–(H3) has the properties

$$\Phi_{11}(a, b) > 0, \quad \Phi_{111}(a, b) < 0, \quad P(a, b) > 0, \tag{3.26}$$

which together with (3.11) imply $\dot{a} > 0, \dot{b} < 0$ and

$$\frac{d}{ds}(a - b) = \frac{1}{sQ}(a - b)[(d - 1)P - Q] > 0.$$

Thus, (3.22), (3.23) and (3.24) must hold for $s \in (0, \varepsilon]$. The solution can be continued in that manner for $s > \varepsilon$ so long as $Q(s) < 0, 0 < a(s) < \infty$, and $0 < b(s) < \infty$ on a maximal interval of existence $[0, T)$, with $T \leq \infty$. It is also clear that the solution cannot hit the diagonal $a = b$ unless $Q = 0$. On the interval (ε, T) we clearly have

$$a(\varepsilon) < a(s) < b(s) < b(\varepsilon), \quad s \in (\varepsilon, T), \tag{3.27}$$

that is $a(s)$ and $b(s)$ stay away from zero and in a bounded range as s increases. Then (3.27) and the fact that $Q(s) = s^2 - \Phi_{11}(a, b) < 0, s \in (0, T)$ imply that $T < \infty$ and

$$\limsup_{s \rightarrow T_-} Q(a(s), b(s), s) = 0. \tag{3.28}$$

Also, in view of (3.22) and (3.23) we have $a - b \rightarrow c_0 \leq 0$ as $s \rightarrow T_-$.

A computation shows that

$$\frac{1}{2} \frac{d}{ds} Q^2 = 2sQ - \frac{d-1}{s} (a-b) [P(a, b)\Phi_{111}(a, b) + \Phi_{112}(a, b)Q].$$

Using (3.12) and the bounds (3.27) we see that for $s \in (\varepsilon, T)$

$$\left| \frac{d}{ds} Q^2 \right| \leq C, \quad s \in (\varepsilon, T). \tag{3.29}$$

Now if $T < \infty$ and

$$\liminf_{s \rightarrow T_-} Q(a(s), b(s), s) < \limsup_{s \rightarrow T_-} Q(a(s), b(s), s) = 0,$$

then (3.29) would be violated. We thus deduce that $Q(s) \rightarrow 0$ as $s \rightarrow T_-$. □

3.2 Connection to a Uniformly Deformed State

Next, the smooth cavitating solution $(\dot{\varphi}, \frac{\varphi}{s})$ of (3.11) constructed in Theorem 3.3 will be connected to a uniformly deformed state through an outgoing shock of speed $\sigma > 0$. At the connection, the Rankine-Hugoniot relations (3.7) must be satisfied and they imply that

$$\varphi_-(\sigma) = \varphi_+(\sigma), \quad \sigma^2(\dot{\varphi}_+(\sigma) - \dot{\varphi}_-(\sigma)) = \Phi_1\left(\dot{\varphi}_+, \frac{\varphi_+}{\sigma}\right) - \Phi_1\left(\dot{\varphi}_-, \frac{\varphi_-}{\sigma}\right),$$

where $\varphi_-(s) \equiv \varphi(s)$ is the given cavitating solution on the left, and $\varphi_+(s) = \lambda s$, for some λ , is the uniformly deformed state on the right.

Thus to connect the cavitating solution $\varphi(s)$ to a uniformly deformed state through a shock wave it suffices to define the function

$$p(s) := \frac{\Phi_1(\dot{\varphi}, \frac{\varphi}{s}) - \Phi_1(\frac{\varphi}{s}, \frac{\varphi}{s})}{\dot{\varphi} - \frac{\varphi}{s}} - s^2 \tag{3.30}$$

and to identify a zero of $p(s)$ for $s \in (0, T)$ the maximal interval of existence of φ . It is expedient to view the right hand side of (3.30) as a function of the principal stretches,

$$R(a, b, s) := \begin{cases} \frac{\Phi_1(a, b) - \Phi_1(b, b)}{a - b} - s^2, & a < b, \\ \Phi_{11}(b, b) - s^2, & a = b, \end{cases} \tag{3.31}$$

and to observe that $R(a, b, s)$ is continuous on $\{(a, b) \in \mathbb{R}^2 : 0 < a \leq b\} \times \mathbb{R}$.

By (3.23) and (3.25), the solutions (v, φ) in Theorem 3.3 have well-defined limits

$$0 < A := \lim_{s \rightarrow T_-} \dot{\varphi}(s) \leq B := \lim_{s \rightarrow T_-} \left(\frac{\varphi}{s}\right) < \infty. \tag{3.32}$$

Theorem 3.4 (Existence of connection point) *Assume $d \geq 2$ and Φ satisfies (H0)–(H4). Let (v, φ) be the solution to (3.14) constructed in Theorem 3.3 and $(a, b) = (\dot{\varphi}, \frac{\varphi}{s})$. Let A, B be the limits defined in (3.32). Then,*

- (i) *if $A = B$ then (a, b) is connected continuously at $\sigma = T$ to the state (B, B) associated with the uniform deformation $\varphi_+(s) = Bs$;*
- (ii) *if $A < B$ then (a, b) is connected to a uniformly deformed state through a Lax-admissible shock at some intermediate point $\sigma \in (0, T)$.*

Proof Recalling (3.31), we set for $s \in (0, T)$

$$\begin{aligned}
 p(s) &:= R\left(\dot{\varphi}, \frac{\varphi}{s}, s\right) \\
 &= \left(\frac{g'(\dot{\varphi}) - g'(\frac{\varphi}{s})}{\dot{\varphi} - \frac{\varphi}{s}}\right) + \left(\frac{\varphi}{s}\right)^{2d-2} \left(\frac{h'(\dot{\varphi}(\frac{\varphi}{s})^{d-1}) - h'((\frac{\varphi}{s})^d)}{\dot{\varphi}(\frac{\varphi}{s})^{d-1} - (\frac{\varphi}{s})^d}\right) - s^2. \tag{3.33}
 \end{aligned}$$

We observe that by (H1), (H4), (3.21), (3.23)–(3.25), (3.32) and (3.33):

- (1) As $s \rightarrow 0+$

$$p(s) \sim \left(\frac{s}{\varphi}\right) g'\left(\frac{\varphi}{s}\right) + \left(\frac{\varphi}{s}\right)^{d-2} h'\left(\left(\frac{\varphi}{s}\right)^d\right) \rightarrow +\infty. \tag{3.34}$$

- (2) As $s \rightarrow T-$

$$p(s) \rightarrow R(A, B, T), \tag{3.35}$$

$$Q(s) \rightarrow T^2 - \Phi_{11}(A, B) = 0. \tag{3.36}$$

Now, denote $p(T) := \lim_{s \rightarrow T-} p(s) = R(A, B, T)$. If $A = B$, then by (3.36)

$$p(T) = R(B, B, T) = \Phi_{11}(B, B) - T^2 = 0.$$

On the other hand, if $A < B$, then (H2) implies $\Phi_{111} < 0$ and by (3.35), (3.36)

$$\begin{aligned}
 p(T) = R(A, B, T) &= \frac{\Phi_1(A, B) - \Phi_1(B, B)}{A - B} - T^2 \\
 &= \Phi_{11}(C^*, B) - \Phi_{11}(A, B) < 0
 \end{aligned}$$

for some $C^* \in (A, B)$. Then (3.34) implies that there exists $\sigma \in (0, T)$ such that $p(\sigma) = 0$. \square

4 Necessity and Uniqueness of the Precursor Shock

In this section we show that there is uniqueness within the method of construction of cavitating solutions and that (when $d = 2, 3$) the connection must happen through a Lax-admissible shock.

We first prove that there exists at most one point in $(0, T]$ where $p(s)$ (defined in (3.33)) vanishes and thus a unique connection (of the cavitating solution with a uniform deformation) occurs.

Theorem 4.1 *Let $d \geq 2$ and Φ satisfy (H0)–(H2). In Theorem 3.4 there exists a unique $\sigma \in (0, T]$ satisfying $p(\sigma) = 0$.*

- (i) *If $\sigma < T$ then the connection happens via a Lax admissible shock.*
- (ii) *If $\sigma = T$ then the connection occurs via a sonic singularity at which $A = B$.*

Proof Consider $p(s)$ for $s \in (0, T)$. Multiply (3.30) by $(a - b)$ and then differentiate to get

$$\begin{aligned} \dot{p}(a - b) + p(\dot{a} - \dot{b}) &= -(s^2 - \Phi_{11}(a, b))\dot{a} \\ &\quad + (d - 1)(\Phi_{12}(a, b) - \Phi_{12}(b, b))\dot{b} \\ &\quad + (s^2 - \Phi_{11}(b, b))\dot{b} - 2s(a - b). \end{aligned}$$

Recall that

$$(s^2 - \Phi_{11}(a, b))\dot{a} = (d - 1)P(a, b)\dot{b}, \quad \dot{b} = \frac{1}{s}(a - b), \quad s \in (0, T) \tag{4.1}$$

and hence

$$\begin{aligned} \dot{p}(a - b) + p(\dot{a} - \dot{b}) &= -(d - 1)[P(a, b) - \Phi_{12}(a, b) + \Phi_{12}(b, b)]\dot{b} - (s^2 + \Phi_{11}(b, b))\dot{b}. \end{aligned}$$

Then, divide the result by $(a - b) < 0$ and use (4.1) to conclude for $s \in (0, T)$

$$\begin{aligned} \dot{p} + p\left(\frac{\dot{a} - \dot{b}}{a - b}\right) &= -\frac{(d - 1)}{s} \left[\frac{\Phi_1(a, b) - \Phi_2(a, b)}{a - b} + \Phi_{12}(b, b) \right] - (s^2 + \Phi_{11}(b, b))\frac{1}{s} < 0. \end{aligned} \tag{4.2}$$

To obtain the sign on the left hand side of (4.2) we note that by (H0) and (H1)

$$\begin{aligned} &\frac{\Phi_1(a, b) - \Phi_2(a, b)}{a - b} + \Phi_{12}(b, b) \\ &= \frac{g'(a) - g'(b)}{a - b} + b^{d-2}(h'(b^d) - h'(ab^{d-1})) + b^{2d-2}h''(b^d) \\ &= g''(c^*) + b^{2d-3}h''(v^*)(b - a) + b^{2d-2}h''(b^d) > 0 \end{aligned}$$

for some $c^* \in (a, b)$ and $v^* \in (ab^{d-1}, b^d)$.

Let now $\sigma \in (0, T]$ satisfying $p(\sigma) = 0$. There exists at least one such point according to Theorem 3.4. On the other hand, (4.2) implies that at any $\sigma \in (0, T)$ where $p(s)$ vanishes we have that $\dot{p}(\sigma) < 0$. Therefore, there can be at most one such point in $(0, T]$.

The discussion in Sect. 3(c) indicates that the shock will satisfy the Lax shock admissibility condition if $\Phi_{111} < 0$. For stored energies of the type (H0) this amounts to condition (H2). □

We next show that for $d \in \{2, 3\}$ the nontrivial solution (v, φ) in Theorem 3.3 cannot be connected continuously to a uniformly deformed state. In view of Theorem 4.1, this means that the cavitating solution is always associated with a unique Lax-admissible precursor shock for dimensions $d = 2, 3$. (The method of proof breaks down for $d \geq 4$.)

Theorem 4.2 Assume $d \in \{2, 3\}$ and Φ satisfies (H0)–(H4). Let (v, φ) be the solution to (3.14) constructed in Theorem 3.3 and let $(a, b) = (\dot{\varphi}, \frac{\varphi}{s})$. Let A, B be the limits defined in (3.32). Then $A < B$ and, as a consequence,

- (i) (a, b) , the solution to (3.11), cannot be connected continuously to a uniformly deformed state at any $s \in (0, T]$;
- (ii) there exists unique point $\sigma \in (0, T)$ at which the solution (a, b) can be connected to a uniformly deformed state associated with $\varphi_+(s) = b(\sigma)s$ through a Lax-admissible shock.

Proof The proof utilizes the properties (7.11) and (7.12) of the function $P(a, b)$ proved in the appendix. Since $Q < 0, s \in (0, T)$, we rewrite (3.11) in the equivalent form

$$\dot{a} = \frac{(d-1)}{s} P(a, b)H(s), \quad \dot{b} = \frac{1}{s}(a-b), \tag{4.3}$$

where the function $H(s)$ is defined by

$$H := \frac{a-b}{Q} = \frac{a-b}{s^2 - \Phi_{11}(a, b)}, \quad s \in (0, T). \tag{4.4}$$

To prove that $A < B$, we will argue by contradiction. Suppose that

$$A = B =: \lambda. \tag{4.5}$$

Step 1. We will prove that (4.5) implies that there exists $\varepsilon > 0$ such that

$$H(s) > 0, \quad \frac{d}{ds}(s^3 H(s)) > 0, \quad s \in (T - \varepsilon, T), \tag{4.6}$$

and, as a consequence, we must have *either*

$$\begin{aligned} \lim_{s \rightarrow T^-} H(s) &= c \quad \text{for some } c < \infty, \\ \text{or} & \\ \lim_{s \rightarrow T^-} H(s) &= +\infty. \end{aligned} \tag{4.7}$$

Indeed, by (3.24) we have $H(s) > 0$ for $s \in (0, T)$. Next, by (3.12), (4.3) we have

$$\begin{aligned} \dot{Q} &= 2s - [\Phi_{111}(a, b)\dot{a} + (d-1)\Phi_{112}\dot{b}] \\ &= 2s - (d-1)H[\Phi_{111}(a, b)P(a, b) + \Phi_{112}(a, b)Q]\frac{1}{s} \\ &= [Q + \Phi_{11}(a, b)]\frac{2}{s} - (d-1)H[\Phi_{111}(a, b)P(a, b) + \Phi_{112}(a, b)Q]\frac{1}{s} \end{aligned} \tag{4.8}$$

and using (4.3), (4.4) we obtain

$$\begin{aligned} \frac{dH}{ds} &= \frac{\dot{a} - \dot{b}}{Q} - \frac{\dot{Q}(a-b)}{Q^2} \\ &= \frac{(d-1)}{s} \frac{H}{Q} P(a, b) - \frac{H}{s} - \frac{H}{Q} \dot{Q} \end{aligned}$$

$$\begin{aligned}
 &= (d - 1) \frac{H}{Q} \left[P(a, b) - \frac{2}{d - 1} \Phi_{11}(a, b) \right] \frac{1}{s} \\
 &\quad + (d - 1) \frac{H^2}{Q} [\Phi_{111}(a, b)P(a, b) + \Phi_{112}(a, b)Q] \frac{1}{s} - \frac{3H}{s}.
 \end{aligned}$$

Rearranging the terms in the resulting expression, we obtain

$$\begin{aligned}
 &\frac{1}{s^2(d - 1)} \frac{d}{ds} (s^3 H) \\
 &= \frac{H}{Q} \left[P(a, b) - \frac{2}{d - 1} \Phi_{11}(a, b) \right] + \frac{H^2}{Q} [\Phi_{111}(a, b)P(a, b) + \Phi_{112}(a, b)Q] \\
 &= H^2 \left(\frac{P(a, b) - \Phi_{11}(a, b)}{a - b} + \Phi_{112}(a, b) \right. \\
 &\quad \left. + \frac{1}{Q} \left[\Phi_{111}(a, b)P(a, b) - \frac{1}{H} \left(\frac{3 - d}{d - 1} \right) \Phi_{11}(a, b) \right] \right). \tag{4.9}
 \end{aligned}$$

Now, observe that (7.11), (3.32), and the assumption (4.5) imply

$$\lim_{s \rightarrow T_-} (\Phi_{111}(a, b)P(a, b)) = \Phi_{111}(\lambda, \lambda)\Phi_{11}(\lambda, \lambda) < 0. \tag{4.10}$$

Recall that $H > 0$. Hence by (3.24), (3.25)₁, (3.26)₁, (4.10) and $d \in \{2, 3\}$ we have

$$\lim_{s \rightarrow T_-} \frac{1}{Q} \left[\Phi_{111}(a, b)P(a, b) - \frac{1}{H} \left(\frac{3 - d}{d - 1} \right) \Phi_{11}(a, b) \right] = +\infty. \tag{4.11}$$

Also, by (3.32), (4.5), and (7.12)

$$\lim_{s \rightarrow T_-} \left(\frac{P(a, b) - \Phi_{11}(a, b)}{a - b} + \Phi_{112}(a, b) \right) = \frac{1}{2} (3\Phi_{112}(\lambda, \lambda) - \Phi_{111}(\lambda, \lambda)). \tag{4.12}$$

Finally, combining (4.11), (4.12), we conclude that there exists $\varepsilon > 0$ such that the right-hand side of (4.9) is positive for all $s \in (T - \varepsilon, T)$. This establishes (4.6) and as a consequence (4.7).

Step 2. By (3.25)₁ and (4.5), both the numerator and the denominator of $H(s)$ satisfy

$$\lim_{s \rightarrow T_-} (a(s) - b(s)) = \lim_{s \rightarrow T_-} Q(s) = 0. \tag{4.13}$$

This motivates to consider the ratio

$$\begin{aligned}
 r(s) &:= \frac{\frac{d}{ds} Q}{\frac{d}{ds} (a - b)} \\
 &= \frac{2s - (d - 1)H[\Phi_{111}(a, b)P(a, b) + \Phi_{112}(a, b)Q] \frac{1}{s}}{(d - 1)HP(a, b) \frac{1}{s} - (a - b) \frac{1}{s}} \\
 &= \frac{\frac{2}{d-1} \frac{s^2}{H} - [\Phi_{111}(a, b)P(a, b) + \Phi_{112}(a, b)Q]}{P(a, b) - \frac{1}{d-1} Q} \tag{4.14}
 \end{aligned}$$

(where we used (4.3), (4.4), (4.8)) and to look for the limiting value of $r(s)$ as $s \rightarrow T_-$. To this end we will use (4.7) and consider two separate cases:

Case 1. Suppose that (4.7)₁ holds. Observe that by (3.25)₁

$$0 = \lim_{s \rightarrow T_-} Q(s) = \Phi_{11}(\lambda, \lambda) - T^2$$

and hence by (7.11) and (4.5)

$$\lim_{s \rightarrow T_-} P(a, b) = T^2.$$

Thus, (3.25)₁, (3.32), (4.5), and (4.10) imply

$$\lim_{s \rightarrow T^-} r(s) = \frac{2}{d-1} \frac{1}{c} - \Phi_{111}(\lambda, \lambda).$$

On the other hand, by l'Hôpital rule, we have

$$0 < \frac{1}{c} = \lim_{s \rightarrow T_-} \frac{1}{H(s)} = \lim_{s \rightarrow T_-} r(s) = \frac{2}{d-1} \frac{1}{c} - \Phi_{111}(\lambda, \lambda).$$

Since $c \in (0, \infty)$ and $d \in \{2, 3\}$ this leads to a contradiction

$$0 < -\Phi_{111}(\lambda, \lambda)c = \left(\frac{d-3}{d-1}\right) \leq 0.$$

Hence (4.7)₁ cannot hold.

Case 2. Suppose that (4.7)₂ holds. Then (3.25)₁, (3.32) and (4.5) imply

$$\lim_{s \rightarrow T^-} r(s) = -\Phi_{111}(\lambda, \lambda). \tag{4.15}$$

By (3.26)₂, (4.4), (4.7)₂, (4.13), (4.14), (4.15) and l'Hopital's rule we obtain

$$0 = \lim_{s \rightarrow T^-} \frac{1}{H(s)} = \lim_{s \rightarrow T^-} r(s) = -\Phi_{111}(\lambda, \lambda) > 0,$$

which is again a contradiction. Thus, (4.7)₂ cannot hold.

Since the hypothesis (4.5) leads to a contradiction, we conclude that $A < B$ for $d = 2, 3$ and the proof is completed. □

5 Rescaling, Inner Solution, and the Dynamic Bifurcation Diagram

The objective of this section is to construct the bifurcation diagram of the dynamically cavitating solution and to determine the critical stretching at which cavitation occurs. We first present an outline of the approach that we follow.

We use the cavitating solution constructed in Theorem 3.4 under hypotheses (H0)–(H3) and hypothesis (H4) (relating to the boundary data at the cavity). Recall that this solution, denoted by $(\varphi, v)(s; \varphi_0, v_0)$, depends on two parameters $\varphi_0 > 0$ and v_0 in (3.14), and it is defined on a maximal interval of existence $[0, T)$ with $T = T(\varphi_0, v_0) < \infty$. By Theorem 4.1 for every pair $\varphi_0, v_0 > 0$ there exists a unique point

$$\sigma = \sigma(\varphi_0, v_0) \in (0, T(\varphi_0, v_0)] \quad \text{that satisfies } p(\sigma; \varphi_0, v_0) = 0, \tag{5.1}$$

where p is defined in (3.30) and the identity $p(\sigma) = 0$ corresponds to the Rankine-Hugoniot jump conditions. We recall that if $\sigma(\varphi_0, v_0) < T(\varphi_0, v_0)$ then the connection of the cavitating solution to the uniformly deformed state on the right happens through a shock, while if $\sigma(\varphi_0, v_0) = T(\varphi_0, v_0)$ then the cavitating solution connects to a uniform deformation in a C^1 fashion through a sonic singularity. Also, that the latter possibility is excluded in Theorem 4.2 for dimensions $d = 2, 3$, but it might conceivably occur for higher dimensions. In both cases, the transversal principal stretch $\frac{\varphi(\sigma)}{\sigma}$ at the shock (or sonic singularity) coincides with the value of external stretching associated to the forming cavity.

In view of the above, we define the mapping $\Lambda(\varphi_0, v_0) : \mathbb{R}_+^2 \rightarrow \mathbb{R}$ by

$$\Lambda(\varphi_0, v_0) = \frac{\varphi(\sigma(\varphi_0, v_0); \varphi_0, v_0)}{\sigma(\varphi_0, v_0)}. \tag{5.2}$$

Referring to the discussion of Sect. 3.1, we recall that at the cavity the specific volume v_0 is connected with the cavity velocity φ_0 via the relation $v_0 = V(\varphi_0)$, where $V : [0, \infty) \rightarrow (0, \infty)$ is a continuous function that encodes the boundary condition at the cavity and has the form

$$V(x) := \begin{cases} H = h'^{-1}(0) & \text{for stress free cavities,} \\ h'^{-1}(G(x)) & \text{for cavities with content.} \end{cases} \tag{5.3}$$

The diagram of the map

$$\varphi_0 \mapsto \Lambda(\varphi_0, V(\varphi_0))$$

is precisely the bifurcation diagram of a solution with cavity. A numerical computation of this diagram appears in Fig. 2; for explanation of numerical results see Appendix 7.3. In the sequel, we study various analytical features of the bifurcation diagram. We are particularly interested in determining the critical stretching λ_{cr} at which a cavity opens. This will be captured by the limit

$$\lim_{\varphi_0 \rightarrow 0^+} \Lambda(\varphi_0, V(\varphi_0)) \tag{5.4}$$

and one objective is to determine a formula for the computation of (5.4) in the dynamic case.

5.1 Rescaling

Let $(\varphi, v)(s; \varphi_0, v_0)$ be the solution to (3.14), with $\varphi_0, v_0 > 0$. Numerical experiments indicate that the functions $\{v(s; \varphi_0, V(\varphi_0))\}$, with V defined by (5.3), converge pointwise to a step function with a jump located at $s = 0$ as $\varphi_0 \rightarrow 0$; see Fig. 1. This indicates an inner layer with respect to φ_0 and points to resolving the jump as the key in determining the limit (5.4).

To capture the behavior near the origin we rescale $(\varphi, v)(s; \varphi_0, v_0)$ with respect to the initial value $\varphi_0 > 0$ using the scaling transformation

$$\psi(\xi; \varphi_0, v_0) := \frac{\varphi(\varphi_0\xi; \varphi_0, v_0)}{\varphi_0}, \quad \delta(\xi; \varphi_0, v_0) := v(\varphi_0\xi; \varphi_0, v_0). \tag{5.5}$$

Note that (ψ, δ) satisfy

$$\frac{\psi(\xi)}{\xi} = \frac{\varphi(s)}{s} \Big|_{s=\varphi_0\xi}, \quad \psi'(\xi) = \dot{\varphi}(s)|_{s=\varphi_0\xi}, \quad \delta(\xi) = v(s)|_{s=\varphi_0\xi},$$

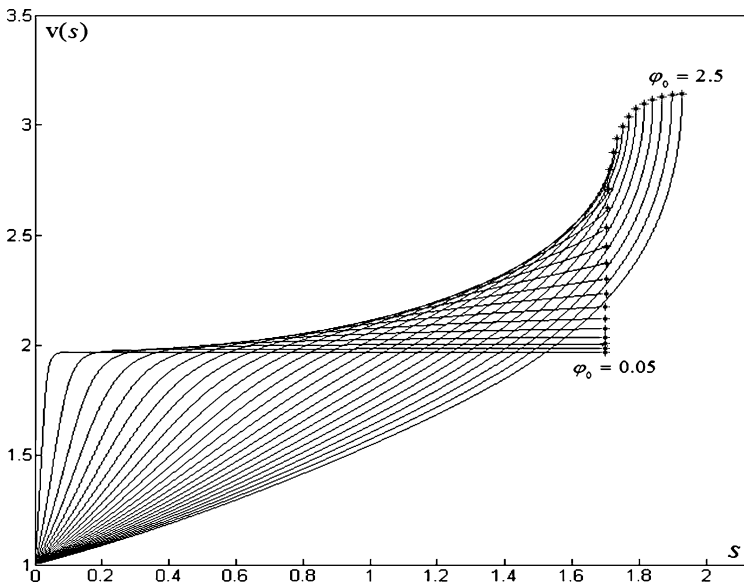


Fig. 1 $v(s; \varphi_0, H)$ with $\varphi_0 \in [0.05, 2.5]$, $g(x) = \frac{1}{2}x^2$, $h(x) = (x - 1) \ln(x)$ and stress free cavity

with $'$ denoting the differentiation with respect to $\xi = \frac{s}{\varphi_0}$. The rescaled function (ψ, δ) is now defined on the maximal interval of existence $[0, \mathcal{T}(\varphi_0, v_0))$, where $\mathcal{T}(\varphi_0, v_0) := \frac{1}{\varphi_0}T(\varphi_0, v_0) < \infty$ and satisfies the initial value problem

$$\left\{ \begin{array}{l} \psi'(\xi) = \delta \left(\frac{\xi}{\psi} \right)^{d-1}, \\ \delta'(\xi) = \left(\frac{d-1}{\psi} \right) \frac{\left(\frac{\xi}{\psi} \right)^{2d-3} \delta \left(\delta \left(\frac{\xi}{\psi} \right)^d - 1 \right) [\xi^2 \varphi_0^2 - g''(\delta \left(\frac{\xi}{\psi} \right)^{d-1})]}{[-h''(\delta) + (\xi^2 \varphi_0^2 - g''(\delta \left(\frac{\xi}{\psi} \right)^{d-1})) \left(\frac{\xi}{\psi} \right)^{2d-2}]} \\ \quad + \left(\frac{d-1}{\psi} \right) \frac{\left(\frac{\xi}{\psi} \right)^{d-2} [g'(\delta \left(\frac{\xi}{\psi} \right)^{d-1}) - g'(\frac{\psi}{\xi})]}{[-h''(\delta) + (\xi^2 \varphi_0^2 - g''(\delta \left(\frac{\xi}{\psi} \right)^{d-1})) \left(\frac{\xi}{\psi} \right)^{2d-2}]}, \\ \psi(0; \varphi_0, v_0) = 1, \\ \delta(0; \varphi_0, v_0) = v_0. \end{array} \right. \tag{5.6}$$

As (3.14) is equivalent to (3.10), the rescaled function $\psi(\xi; \varphi_0, v_0)$ will also satisfy the second order differential equation

$$\left(\varphi_0^2 \xi^2 - \Phi_{11} \left(\psi', \frac{\psi}{\xi} \right) \right) \psi'' = \frac{d-1}{\xi} \left(\psi' - \frac{\psi}{\xi} \right) P \left(\psi', \frac{\psi}{\xi} \right). \tag{5.7}$$

Theorem 3.3 now gives:

Lemma 5.1 *Let $d \geq 2$ and Φ satisfy (H0)–(H3). The rescaled function (ψ, δ) in (5.5) is defined on the maximal interval of existence $[0, \mathcal{T}(\varphi_0, v_0))$, with $\mathcal{T}(\varphi_0, v_0) := \frac{1}{\varphi_0}T(\varphi_0, v_0)$, and satisfies (5.6), while $\psi \in C^2(0, \mathcal{T})$ solves (5.7) on $(0, \mathcal{T})$. Moreover,*

(i) $\psi', \frac{\psi}{\xi}, (\psi' - \frac{\psi}{\xi})$ are strictly monotonic and satisfy

$$\psi''(\xi) > 0, \quad \frac{d}{d\xi} \left(\frac{\psi}{\xi} \right) < 0, \quad \frac{d}{d\xi} \left(\psi' - \frac{\psi}{\xi} \right) > 0 \quad \text{on } 0 < \xi < \mathcal{T}. \tag{5.8}$$

(ii) For each $0 < \xi < \tau < \mathcal{T}$

$$0 < \psi'(\xi) < \psi'(\tau) < \frac{\psi(\tau)}{\tau} < \frac{\psi(\xi)}{\xi}, \quad Q \left(\psi', \frac{\psi}{\xi}, \xi \varphi_0 \right) < 0. \tag{5.9}$$

(iii) The following limits exists and satisfy

$$\lim_{\xi \rightarrow \mathcal{T}_-} Q \left(\psi', \frac{\psi}{\xi}, \xi \varphi_0 \right) = 0, \quad \lim_{\xi \rightarrow \mathcal{T}_-} \psi' = A \leq \lim_{\xi \rightarrow \mathcal{T}_-} \left(\frac{\psi}{\xi} \right) = B$$

for some $A, B > 0$.

Due to the form of (5.7) the rescaling leads to a regular perturbation problem and it is expected that the limit has a globally defined solution. Below, we study this limiting process.

5.2 Uniform Bounds for the Rescaled Solutions

In the sequel we impose the condition that for some $\nu > 0$

$$\Phi_{11}(x, x) = g''(x) + x^{2d-2}h''(x^d) \geq \nu^2 > 0, \quad \text{for } 0 < x < \infty. \tag{H5}$$

This condition is fulfilled if g satisfies $g''(x) \geq \nu^2 > 0$. The latter assumption is possible for $d \geq 3$, but it is inconsistent with (H3) for $d = 2$. An alternative is to impose at infinity the condition

$$\limsup_{x \rightarrow \infty} h''(x)x^{2-\frac{2}{d}} > 0, \tag{H5'}$$

which together with (H1) implies (H5). Hypothesis (H5) ensures that $\mathcal{T}(\varphi_0, \nu_0) \rightarrow \infty$ as $\varphi_0 \rightarrow 0_+$.

Lemma 5.2 *Let Φ satisfy (H0)–(H3), (H5). Let $T(\varphi_0, \nu_0), \mathcal{T}(\varphi_0, \nu_0)$ denote the times of existence defined in Theorem 3.3 and Lemma 5.1, respectively. Then,*

$$T(\varphi_0, \nu_0) = \varphi_0 \mathcal{T}(\varphi_0, \nu_0) > \nu, \quad \varphi_0, \nu_0 > 0. \tag{5.10}$$

Proof By (H2), (3.26) and (H5) we get

$$\Phi_{11}(a, b) > \Phi_{11}(b, b) > \nu^2 > 0 \tag{5.11}$$

for all $0 < a \leq b$. Let $(\varphi, \nu)(s; \varphi_0, \nu_0)$ be the solution of (3.14) and A, B the limits defined in (3.32). Then from (3.12), (3.25), (3.32) and (5.5) we conclude

$$\varphi_0 \mathcal{T}(\varphi_0, \nu_0) = T(\varphi_0, \nu_0) = \sqrt{\Phi_{11}(A, B)} > \nu. \quad \square$$

In the sequel we employ the notation

$$f(\xi; \varphi_0, \nu_0) = \Phi_{11} \left(\psi', \frac{\psi}{\xi} \right) \left(\frac{\xi}{\psi} \right)^{2d-2} \tag{5.12}$$

$$= h''(\delta) + g''\left(\delta\left(\frac{\xi}{\psi}\right)^{d-1}\right)\left(\frac{\xi}{\psi}\right)^{2d-2}, \tag{5.13}$$

$$\widehat{Q}(\xi; \varphi_0, v_0) = \xi^2 \varphi_0^2 - \Phi_{11}\left(\psi', \frac{\psi}{\xi}\right) \tag{5.14}$$

$$= \xi^2 \varphi_0^2 - \left[g''\left(\delta\left(\frac{\xi}{\psi}\right)^{d-1}\right) + \left(\frac{\psi}{\xi}\right)^{2d-2} h''(\delta) \right], \tag{5.15}$$

$$D(\xi; \varphi_0, v_0) = \left(\frac{\xi}{\psi}\right)^{2d-3} \delta\left(\delta\left(\frac{\xi}{\psi}\right)^d - 1\right)\left(\xi^2 \varphi_0^2 - g''\left(\delta\left(\frac{\xi}{\psi}\right)^{d-1}\right)\right) + \left(\frac{\xi}{\psi}\right)^{d-2} \left(g'\left(\delta\left(\frac{\xi}{\psi}\right)^{d-1}\right) - g'\left(\frac{\psi}{\xi}\right)\right), \tag{5.16}$$

where $(\psi, \delta)(\xi; \varphi_0, v_0)$ is the solution of (5.6). We note that under Hypothesis (H3)

$$|g'(x)| \leq \widehat{\gamma} \max(1, x^{d-2}), \quad \text{for } x \in [0, \infty). \tag{5.17}$$

The next lemma provides bounds on the Cauchy stress.

Lemma 5.3 *Assume $d \geq 2$ and Φ satisfies (H0)–(H3), (H5). Let $(\psi, \delta)(\xi; \varphi_0, v_0)$ be the solution to (5.6) defined on a maximal interval of existence $[0, \mathcal{T}(\varphi_0, v_0))$. Suppose $\tau > 0$ is fixed, $v > 0$ satisfies (5.11) and*

$$\varepsilon_\tau := v(2(1 + v + v^{-1} + \tau))^{-1}. \tag{5.18}$$

Then, for every

$$0 < \varphi_0 < \varepsilon_\tau, \quad 0 < v_0 < \infty,$$

the interval $[0, \tau] \subset [0, \mathcal{T}(\varphi_0, v_0))$ and

$$\begin{aligned} \widehat{T}_{rad}(\xi; \varphi_0, v_0) &:= \left[\Phi_1\left(\psi', \frac{\psi}{\xi}\right)\left(\frac{\xi}{\psi}\right)^{d-1} \right](\xi; \varphi_0, v_0) \\ &= \left[h'(\delta) + g'\left(\delta\left(\frac{\xi}{\psi}\right)^{d-1}\right)\left(\frac{\xi}{\psi}\right)^{d-1} \right](\xi; \varphi_0, v_0) \end{aligned} \tag{5.19}$$

satisfies

$$\left| \frac{d}{d\xi}(\widehat{T}_{rad}(\xi; \varphi_0, v_0)) \right| < c_{rad}(1 + \tau^{d+1}), \quad \xi \in (0, \tau] \tag{5.20}$$

with $c_{rad} := 6(d - 1)(1 + g''(0) + \widehat{\gamma})$.

Proof Take $\varphi_0 \in (0, \varepsilon_\tau)$, $v_0 \in (0, \infty)$. By (5.10) the maximal time $\mathcal{T}(\varphi_0, v_0) > \frac{v}{\varphi_0}$ thus enforcing that $[0, \tau] \subset [0, \mathcal{T}(\varphi_0, v_0))$ and $\widehat{T}_{rad}(\xi; \varphi_0, v_0)$ is well-defined on $(0, \tau]$.

Next, we compute

$$\frac{d}{d\xi}(\widehat{T}_{rad}(\xi; \varphi_0, v_0)) = \frac{d}{d\xi}\left(g'\left(\delta\left(\frac{\xi}{\psi}\right)^{d-1}\right)\left(\frac{\xi}{\psi}\right)^{d-1} + h'(\delta)\right)$$

$$\begin{aligned}
 &= \left(g'' \left(\delta \left(\frac{\xi}{\psi} \right)^{d-1} \right) \left(\frac{\xi}{\psi} \right)^{2d-2} + h''(\delta) \right) \delta' \\
 &\quad + \frac{d-1}{\psi} \left(\frac{\xi}{\psi} \right)^{2d-3} \delta \left(1 - \delta \left(\frac{\xi}{\psi} \right)^d \right) g'' \left(\delta \left(\frac{\xi}{\psi} \right)^{d-1} \right) \\
 &\quad + \frac{d-1}{\psi} \left(\frac{\xi}{\psi} \right)^{d-2} \left(1 - \delta \left(\frac{\xi}{\psi} \right)^d \right) g' \left(\delta \left(\frac{\xi}{\psi} \right)^{d-1} \right). \tag{5.21}
 \end{aligned}$$

Observe that by (5.6)₂, (5.16), and (5.15)

$$\left(g'' \left(\delta \left(\frac{\xi}{\psi} \right)^{d-1} \right) \left(\frac{\xi}{\psi} \right)^{2d-2} + h''(\delta) \right) \delta' = \frac{d-1}{\psi} D(\xi) \left(\frac{\xi^2 \varphi_0^2}{Q(\xi)} - 1 \right)$$

and hence (5.21) reads

$$\begin{aligned}
 &\frac{1}{d-1} \frac{d}{d\xi} (\widehat{T}_{rad}(\xi; \varphi_0, v_0)) \\
 &= \left(\frac{\xi}{\psi} \right)^{2d-2} \delta \left(1 - \delta \left(\frac{\xi}{\psi} \right)^d \right) \xi \varphi_0^2 + \left(\frac{\xi}{\psi} \right) \frac{D(\xi)}{Q(\xi)} \xi \varphi_0^2 \\
 &\quad + \frac{1}{\psi} \left(\frac{\xi}{\psi} \right)^{d-2} \left(g' \left(\frac{\psi}{\xi} \right) - \delta \left(\frac{\xi}{\psi} \right)^d g' \left(\delta \left(\frac{\xi}{\psi} \right)^{d-1} \right) \right) \\
 &=: I_1 + I_2 + I_3. \tag{5.22}
 \end{aligned}$$

We now estimate the right-hand side of (5.22). By (5.8)–(5.9)

$$1 = \psi(0) < \psi(\xi), \quad 0 < \left(\frac{\psi' \xi}{\psi} \right) = \delta \left(\frac{\xi}{\psi} \right)^d < 1, \quad \xi \in (0, \mathcal{T}) \tag{5.23}$$

and hence

$$0 \leq I_1 = \left(\frac{\xi}{\psi} \right)^{2d-2} \delta \left(1 - \delta \left(\frac{\xi}{\psi} \right)^d \right) \xi \varphi_0^2 < \xi^{d-1} \varphi_0^2 \leq \tau^{d-1} \varphi_0^2, \quad \xi \in (0, \tau]. \tag{5.24}$$

Next, by (H3), (5.23)₂, and (5.17) obtain

$$\begin{aligned}
 |I_3| &\leq \left| \left(\frac{\xi}{\psi} \right)^{d-2} g' \left(\frac{\psi}{\xi} \right) \right| + \left| \left(\frac{\xi}{\psi} \right)^{d-2} g' \left(\delta \left(\frac{\xi}{\psi} \right)^{d-1} \right) \right| \\
 &= \left| \left(\frac{\xi}{\psi} \right)^{d-2} g' \left(\frac{\psi}{\xi} \right) \right| + \left| \left(\frac{\psi' \xi}{\psi} \right)^{d-2} \frac{g'(\psi')}{(\psi')^{d-2}} \right| \leq 2\widehat{\gamma}(1 + \tau^{d-2}), \quad \xi \in (0, \tau]. \tag{5.25}
 \end{aligned}$$

Now, using (5.23), we obtain the estimate

$$\left| \left(\frac{\xi}{\psi} \right)^{2d-3} \delta \left(\delta \left(\frac{\xi}{\psi} \right)^d - 1 \right) \right| = \left(\frac{\psi}{\xi} \right) \left| \left(\frac{\xi}{\psi} \right)^{d-2} \delta \left(\frac{\xi}{\psi} \right)^d \left(\delta \left(\frac{\xi}{\psi} \right)^d - 1 \right) \right| \leq \left(\frac{\psi}{\xi} \right) \tau^{d-2},$$

which together with (H1), (H2), (5.16), (5.23)₂, and (5.25) gives the estimate

$$\begin{aligned}
 |D(\xi; \varphi_0, v_0)| &\leq \left| \left(\frac{\xi}{\psi} \right)^{2d-3} \delta \left(\delta \left(\frac{\xi}{\psi} \right)^d - 1 \right) \right| (\xi^2 \varphi_0^2 + g''(0)) \\
 &\quad + \left| \left(\frac{\xi}{\psi} \right)^{d-2} g' \left(\delta \left(\frac{\xi}{\psi} \right)^{d-1} \right) \right| + \left| \left(\frac{\xi}{\psi} \right)^{d-2} g' \left(\frac{\psi}{\xi} \right) \right| \\
 &\leq \left(\frac{\psi}{\xi} \right) (1 + \tau^d) (\varphi_0^2 + g''(0)) + 2\widehat{\gamma} (1 + \tau^{d-2}), \quad \xi \in (0, \tau]. \tag{5.26}
 \end{aligned}$$

Next, since $\varphi_0 \in (0, \varepsilon_\tau)$, by (5.11) we have

$$-\widehat{Q}(\xi; \varphi_0, v_0) = \Phi_{11} \left(\psi', \frac{\psi}{\xi} \right) - \varphi_0^2 \xi^2 > \frac{\nu^2}{2}, \quad \xi \in (0, \tau], \tag{5.27}$$

and therefore, employing (5.18), (5.26) and (5.23), we obtain

$$\begin{aligned}
 |I_2| &= \left| \left(\frac{\xi}{\psi} \right) \frac{D(\xi)}{\widehat{Q}(\xi)} \xi \varphi_0^2 \right| \leq \frac{2}{\nu^2} \left((1 + \tau^d) (\varphi_0^2 + g''(0)) + 2\widehat{\gamma} (1 + \tau^{d-2}) \left(\frac{\xi}{\psi} \right) \right) \xi \varphi_0^2 \\
 &\leq \frac{2\varphi_0}{\nu^2} ((\tau + \tau^{d+1}) (\varphi_0^2 + g''(0)) + 2\widehat{\gamma} (\tau^2 + \tau^d)) \varphi_0 \\
 &\leq 4(1 + \tau^{d+1}) (\varphi_0^2 + g''(0) + \widehat{\gamma}) \varphi_0, \quad \xi \in (0, \tau], \tag{5.28}
 \end{aligned}$$

where we used the fact that $\frac{2\varphi_0}{\nu^2} < 1$ because $0 < \varphi_0 < \varepsilon_\tau$. Thus by (5.25) and (5.28) we conclude

$$|I_2| + |I_3| \leq 4(1 + \tau^{d+1}) (\varphi_0^2 + g''(0) + \widehat{\gamma}) \varphi_0 + 2\widehat{\gamma} (1 + \tau^{d-2}), \quad \xi \in (0, \tau]. \tag{5.29}$$

Combining (5.22), (5.24) and (5.29) we obtain (5.20). □

Lemma 5.4 *Assume $d \geq 2$ and Φ satisfies (H0)–(H5). Let $(\psi, \delta)(\xi; \varphi_0, v_0)$ be the solution to (5.6) defined on a maximal interval of existence $[0, \mathcal{T}(\varphi_0, v_0))$. Let $\tau > 0, v_M > v_m > 0$ be fixed. Then, for every*

$$0 < \varphi_0 < \varepsilon_\tau, \quad v_m \leq v_0 \leq v_M,$$

the interval $[0, \tau] \subset [0, \mathcal{T}(\varphi_0, v_0))$ and

$$0 < c_1 < \delta(\xi; \varphi_0, v_0) < c_2, \quad 1 \leq \psi(\xi; \varphi_0, v_0) < c_3, \quad \forall \xi \in [0, \tau]$$

for some $c_1 = c_1(\tau, v_m), c_2 = c_2(\tau, v_M), c_3 = c_3(\tau, v_M)$ independent of φ_0, v_0 .

Proof Fix $\tau > 0, v_M > v_m > 0$. Take any $\varphi_0 \in (0, \varepsilon_\tau), v_0 \in [v_m, v_M]$ and consider the solution $(\psi, \delta)(\xi; \varphi_0, v_0)$ to (5.6). From (5.19) it follows that

$$\widehat{T}_{rad}(0; \varphi_0, v_0) := \lim_{\xi \rightarrow 0^+} \widehat{T}_{rad}(\xi; \varphi_0, v_0) = h'(v_0)$$

and hence, using (5.20), we obtain for $\xi \in [0, \tau]$

$$\begin{aligned} & \left| \widehat{T}_{rad}(\xi; \varphi_0, v_0) - \widehat{T}_{rad}(0; \varphi_0, v_0) \right| \\ &= \left| h'(\delta) + g' \left(\delta \left(\frac{\xi}{\psi} \right)^{d-1} \right) \left(\frac{\xi}{\psi} \right)^{d-1} - h'(v_0) \right| < c_{rad}(1 + \tau^{d+1})\tau. \end{aligned}$$

Then, similarly to (5.25), using (5.17) and (5.23), we get

$$\left| \left(\frac{\xi}{\psi} \right)^{d-1} g' \left(\delta \left(\frac{\xi}{\psi} \right)^{d-1} \right) \right| \leq \widehat{\gamma}(1 + \tau^{d-2})\tau, \quad \xi \in (0, \tau]$$

and therefore

$$\left| h'(\delta(\xi; \varphi_0, v_0)) - h'(v_0) \right| < 2(c_{rad} + \widehat{\gamma})(1 + \tau^{d+2}), \quad \xi \in [0, \tau].$$

By (H1), (H4), the inverse $h^{-1}(z)$ is well-defined for all $z \in (-\infty, \infty)$ and strictly positive. Hence

$$0 < c_1 < \delta(\xi; \varphi_0, v_0) < c_2, \quad \xi \in [0, \tau],$$

with

$$\begin{aligned} c_1(\tau, v_m) &:= h^{-1}(h'(v_m) - 2(c_{rad} + \widehat{\gamma})(1 + \tau^{d+2})), \\ c_2(\tau, v_M) &:= h^{-1}(h'(v_M) + 2(c_{rad} + \widehat{\gamma})(1 + \tau^{d+2})). \end{aligned}$$

Next, by (5.6)₁, (5.23)₁ and the bound for δ we obtain

$$0 < \psi'(\xi; \varphi_0, v_0) = \delta \left(\frac{\xi}{\psi} \right)^{d-1} < \delta \tau^{d-1} < c_2 \tau^{d-1}, \quad \xi \in (0, \tau]$$

and hence, since $\psi(0) = 1$, we conclude

$$1 \leq \psi(\xi; \varphi_0, v_0) < c_3(\tau, v_M) := 1 + c_2 \tau^d, \quad \xi \in [0, \tau]. \quad \square$$

5.3 The Limiting System: $\varphi_0 = 0$

In this section we consider the limiting problem $\varphi_0 = 0$ in (5.6) and discuss its connection to the problem of radial solutions for equilibrium elasticity analyzed by [2].

5.3.1 Inner Solution

Setting $\varphi_0 = 0$ into (5.6) we arrive at the initial value problem

$$\left\{ \begin{aligned} \psi'_0 &= \delta_0 \left(\frac{\xi}{\psi_0} \right)^{d-1}, \\ \delta'_0 &= \left(\frac{d-1}{\psi_0} \right) \frac{-\left(\frac{\xi}{\psi_0}\right)^{2d-3} \delta_0 (\delta_0 (\frac{\xi}{\psi_0})^d - 1) g''(\delta_0 (\frac{\xi}{\psi_0})^{d-1})}{[-h''(\delta_0) - g''(\delta_0 (\frac{\xi}{\psi_0})^{d-1}) (\frac{\xi}{\psi_0})^{2d-2}]} \\ &\quad + \left(\frac{d-1}{\psi_0} \right) \frac{\left(\frac{\xi}{\psi_0}\right)^{d-2} [g'(\delta_0 (\frac{\xi}{\psi_0})^{d-1}) - g'(\frac{\psi_0}{\xi})]}{[-h''(\delta_0) - g''(\delta_0 (\frac{\xi}{\psi_0})^{d-1}) (\frac{\xi}{\psi_0})^{2d-2}]}, \\ \psi_0(0; v_0) &= 1, \\ \delta_0(0; v_0) &= v_0, \end{aligned} \right. \tag{5.30}$$

while setting $\varphi_0 = 0$ into (5.7) gives the second order differential equation

$$-\Phi_{11} \left(\psi'_0, \frac{\psi_0}{\xi} \right) \psi''_0 = \frac{d-1}{\xi} \left(\psi'_0 - \frac{\psi_0}{\xi} \right) P \left(\psi'_0, \frac{\psi_0}{\xi} \right). \tag{5.31}$$

First, we show that (5.30) has a globally-defined solution. This is an *inner solution* that describes the behavior of a material with a cavity whose surface moves with infinitely small speed.

Proposition 5.5 *Let $d \geq 2$, let Φ satisfy (H0)–(H3), and $v_0 > 0$ be fixed. Then, there exists a unique global solution*

$$\psi_0(\xi; v_0) \in C^2[0, \infty), \quad \delta_0(\xi; v_0) \in C^1[0, \infty),$$

to the initial value problem (5.30) with ψ_0 solving Eq. (5.31) on $(0, \infty)$. The solution has the following properties:

(i) $\psi'_0, \frac{\psi_0}{\xi}, (\psi'_0 - \frac{\psi_0}{\xi})$ are strictly monotonic and satisfy

$$\psi''_0(\xi) > 0, \quad \frac{d}{d\xi} \left(\frac{\psi_0}{\xi} \right) < 0, \quad \frac{d}{d\xi} \left(\psi'_0 - \frac{\psi_0}{\xi} \right) > 0, \quad \xi \in (0, \infty). \tag{5.32}$$

$$0 < \psi'_0(\xi) < \psi'_0(\tau) < \frac{\psi_0(\tau)}{\tau} < \frac{\psi_0(\xi)}{\xi}, \quad 0 < \xi < \tau < \infty. \tag{5.33}$$

(ii) The following limits exist and satisfy

$$0 < \lim_{\xi \rightarrow \infty} \psi'_0 = \lim_{\xi \rightarrow \infty} \left(\frac{\psi_0}{\xi} \right) =: \Lambda_0(v_0) < \infty \quad \text{for some } \Lambda_0 = \Lambda_0(v_0) > 0. \tag{5.34}$$

(iii) ψ_0, δ_0 obey the bounds

$$\max(1, \Lambda_0 \xi) < \psi_0(\xi) < 1 + \Lambda_0 \xi, \quad 0 < \psi'_0(\xi) < \Lambda_0, \tag{5.35}$$

$$0 < \mu_1 < \delta_0(\xi) < \mu_2, \tag{5.36}$$

for $\xi \in [0, \infty)$ and some $\mu_1 = \mu_1(v_0), \mu_2 = \mu_2(v_0)$. In addition,

$$\delta'_0(\xi) \in L^1(0, \infty). \tag{5.37}$$

Proof The first part of the proof proceeds along the lines of the proof of Theorem 3.3. Fix $v_0 > 0$. Using (H3), there exists a unique local solution (ψ_0, δ_0) of (5.30). Clearly, ψ_0 satisfies (5.31), and if we set

$$a_0 = \psi'_0, \quad b_0 = \frac{\psi_0}{\xi},$$

(a_0, b_0) also satisfies the system of ordinary differential equations

$$\begin{aligned} a'_0 &= -\frac{d-1}{\xi}(a_0 - b_0) \frac{P(a_0, b_0)}{\Phi_{11}(a_0, b_0)}, \\ b'_0 &= \frac{1}{\xi}(a_0 - b_0). \end{aligned} \tag{5.38}$$

By construction, the solution (a_0, b_0) starts above the diagonal line of equilibria $a_0 = b_0$ for (5.38). Using (3.26) we see that the monotonicity properties (5.32) and (5.33) hold, that the solution (a_0, b_0) is globally defined, and that the following limits exist

$$\lim_{\xi \rightarrow \infty} \psi'_0 = A \leq \lim_{\xi \rightarrow \infty} \left(\frac{\psi_0}{\xi} \right) = B, \quad \lim_{\xi \rightarrow \infty} \delta_0 = AB^{d-1},$$

for some finite $A, B > 0$.

From (5.38) we deduce that

$$\frac{d}{d\xi}(b_0 - a_0) + \frac{1}{\xi} \left(1 + (d-1) \frac{P(a_0, b_0)}{\Phi_{11}(a_0, b_0)} \right) (b_0 - a_0) = 0$$

and using again (3.26)

$$\begin{aligned} 0 < (b_0 - a_0)(\xi) &= (b_0 - a_0)(1) \exp \left\{ - \int_1^\xi \frac{1}{\zeta} \left(1 + (d-1) \frac{P(a_0, b_0)}{\Phi_{11}(a_0, b_0)} \right) d\zeta \right\} \\ &\leq \frac{(b_0 - a_0)(1)}{\xi}. \end{aligned} \tag{5.39}$$

Hence, $A = B =: \Lambda_0$, and the monotonicity properties (5.32) yield the bounds

$$1 < \psi_0, \quad 0 < \psi'_0 < \Lambda_0, \quad \Lambda_0 \xi < \psi_0 < 1 + \Lambda_0 \xi,$$

thus providing (5.35) and (5.36). Finally, recalling (5.30)₁, we have

$$\delta'_0 = \psi''_0 \left(\frac{\psi_0}{\xi} \right)^{d-1} + (d-1) \psi'_0 \left(\frac{\psi_0}{\xi} \right)^{d-2} \frac{d}{d\xi} \left(\frac{\psi_0}{\xi} \right).$$

Moreover, (5.34), (5.39) and (5.31) imply

$$\left| \frac{d}{d\xi} \left(\frac{\psi_0}{\xi} \right) \right| \leq \frac{C}{\xi^2}, \quad 0 < \psi''_0 \leq \frac{C}{\xi^2}, \quad \text{for } \xi \geq 1,$$

and thus δ'_0 is integrable on $[1, \infty)$. Since $\delta_0 \in C^1[0, \infty)$, we conclude with (5.37). □

For future purposes we introduce the notation

$$f_0(\xi; v_0) = \Phi_{11} \left(\psi'_0, \frac{\psi_0}{\xi} \right) \left(\frac{\xi}{\psi_0} \right)^{2d-2}$$

$$= h''(\delta_0) + g''\left(\delta_0\left(\frac{\xi}{\psi_0}\right)^{d-1}\right)\left(\frac{\xi}{\psi_0}\right)^{2d-2}, \tag{5.40}$$

$$D_0(\xi; v_0) = -\left(\frac{\xi}{\psi_0}\right)^{2d-3} \delta_0\left(\delta_0\left(\frac{\xi}{\psi_0}\right)^d - 1\right) g''\left(\delta_0\left(\frac{\xi}{\psi_0}\right)^{d-1}\right) + \left(\frac{\xi}{\psi_0}\right)^{d-2} \left[g'\left(\delta_0\left(\frac{\xi}{\psi_0}\right)^{d-1}\right) - g'\left(\frac{\psi_0}{\xi}\right) \right]. \tag{5.41}$$

5.3.2 Connections to Equilibrium Elasticity: Representations of Λ_0

In [2] J. Ball considers the boundary-value problem for the equations of equilibrium elasticity

$$\begin{cases} \operatorname{div}\left(\frac{\partial W}{\partial F}(\nabla y)\right) = 0, & x \in \mathcal{B} = \{x \in \mathbb{R}^d : |x| < 1\}, \\ y(x) = \lambda x, & x \in \partial\mathcal{B}, \end{cases} \tag{5.42}$$

for some stretch $\lambda > 0$ and studies radial solutions to (5.42) that have the form

$$y(x) = \frac{w(|x|)}{|x|} \quad \text{with } w(R) : [0, \infty) \rightarrow [0, \infty). \tag{5.43}$$

The amplitude w of such solution satisfies the boundary-value problem

$$\begin{cases} \frac{d}{dR}\left(R^{d-1}\Phi_1\left(w', \frac{w}{R}\right)\right) - (d-1)R^{d-2}\Phi_2\left(w', \frac{w}{R}\right) = 0, \\ w(1) = \lambda. \end{cases} \tag{5.44}$$

It is proved in [2, Theorem 7.9] that a solution w_λ of (5.44) with $w_\lambda(0) > 0$ and $T_{rad}[w_\lambda](0) = 0$ exists if and only if $\lambda > \lambda_{cr}$, where

$$\lambda_{cr} := \lim_{R \rightarrow \infty} \frac{w(R)}{R} \tag{5.45}$$

and $w(R)$ is any solution to radial elastostatics (5.44)₁ with $w(0) > 0$ and $T_{rad}[w](0) = 0$ (such solutions exist by [2, Theorems 7.3, 7.7]); here

$$T_{rad}[w](R) := \Phi_1\left(w', \frac{w}{R}\right)\left(\frac{R}{w}\right)^{d-1}$$

denotes the radial component of the Cauchy stress. Moreover, the solution w_λ generates a cavitating weak solution to (5.42) via the formula (5.43).

Let $(\psi_0, \delta_0)(\xi; v_0)$ be the solution of (5.30). Then, recalling (3.5), observe that (5.31) translates into

$$\frac{d}{d\xi}\left(\xi^{d-1}\Phi_1\left(\psi_0', \frac{\psi_0}{\xi}\right)\right) - (d-1)\xi^{d-2}\Phi_2\left(\psi_0', \frac{\psi_0}{\xi}\right) = 0.$$

Thus, $\psi_0(\xi_0; v_0)$ is a solution to (5.44), with $\lambda = \psi_0(1; v_0)$. By (3.19)

$$T_{rad}[\psi_0(\cdot; v_0)](0) = 0 \quad \text{if and only if } v_0 = H,$$

and hence (5.34), (5.45) imply that the *critical stretch* λ_{cr} for radial equilibrium elasticity (in the stress-free case) satisfies

$$\lambda_{cr} = \Lambda_0(H). \tag{5.46}$$

Representations of Λ_0 We briefly discuss some of the representations of $\Lambda_0(v_0)$ defined in (5.34). Suppose $d \geq 2$, Φ satisfies (H0)–(H3) and $(\psi_0, \delta_0)(\xi; v_0)$ is the solution of (5.30). Then (5.30)₁ and (5.34) imply

$$\delta_0(\xi; v_0) = \left[\psi_0' \left(\frac{\psi_0}{\xi} \right)^{d-1} \right] (\xi; v_0) \rightarrow (\Lambda_0(v_0))^d \quad \text{as } \xi \rightarrow \infty.$$

Thus, integrating (5.30)₂ and recalling (5.37), we conclude

$$\Lambda_0(v_0) = \sqrt[d]{v_0 + \int_0^\infty (1-d) \left[\frac{D_0}{\psi_0 f_0} \right] (\xi; v_0) d\xi}, \tag{5.47}$$

with f_0, D_0 defined by (5.40), (5.41). Since the limiting system (5.30) is non-singular, the integral in (5.47) is well-defined; its rate of convergence is $O(\xi^{-1})$.

Another representation can be obtained by differentiating

$$T_{rad}[\psi_0(\xi; v_0)] = \left[h'(\delta_0) + g' \left(\delta_0 \left(\frac{\xi}{\psi_0} \right)^{d-1} \right) \left(\frac{\xi}{\psi_0} \right)^{d-1} \right] (\xi; v_0),$$

which, in view of (5.30)₂, gives

$$\frac{d}{d\xi} (T_{rad}[\psi_0(\xi; v_0)]) = \left\{ \frac{d-1}{\psi_0} \left(\frac{\xi}{\psi_0} \right)^{d-2} \left[g' \left(\frac{\psi_0}{\xi} \right) - \delta \left(\frac{\xi}{\psi_0} \right)^d g' \left(\delta_0 \left(\frac{\xi}{\psi_0} \right)^{d-1} \right) \right] \right\} (\xi; v_0). \tag{5.48}$$

(We note that (5.48) can be directly obtained by setting $\varphi_0 = 0$ in (5.22).) Integrating the above identity over $(0, \xi)$, letting $\xi \rightarrow \infty$ and using (5.34) we find

$$\begin{aligned} &\chi(\Lambda_0(v_0)) \\ &= h'(v_0) + \int_0^\infty \left\{ \frac{d-1}{\psi_0} \left(\frac{\xi}{\psi_0} \right)^{d-2} \left[g' \left(\frac{\psi_0}{\xi} \right) - \delta_0 \left(\frac{\xi}{\psi_0} \right)^d g' \left(\delta_0 \left(\frac{\xi}{\psi_0} \right)^{d-1} \right) \right] \right\} (\xi; v_0) d\xi, \end{aligned}$$

where

$$\chi(x) := \Phi_1(x, x)x^{1-d} = h'(x^d) + g'(x)x^{1-d}, \quad x \in (0, \infty).$$

Suppose that

$$\chi'(x) > 0 \quad \text{and} \quad \limsup_{x \rightarrow 0_+} (h'(x)x) < 0. \tag{H6}$$

In that case, the inverse χ^{-1} is well-defined and hence

$$\begin{aligned} \Lambda_0(v_0) &= \chi^{-1} \left(h'(v_0) + \int_0^\infty \left[\frac{d-1}{\xi} \left(\frac{\xi}{\psi_0} \right)^{d-2} \right. \right. \\ &\quad \left. \left. \times \left(g' \left(\frac{\psi_0}{\xi} \right) - \delta_0 \left(\frac{\xi}{\psi_0} \right)^d g' \left(\delta_0 \left(\frac{\xi}{\psi_0} \right)^{d-1} \right) \right) \right] (\xi; v_0) d\xi \right). \end{aligned} \tag{5.49}$$

For further ideas on the representation of $\Lambda_0(v_0)$ we refer the reader to [2].

From the representation formulas (5.47), (5.49) one can derive lower bounds for the critical stretch for cavitation $\Lambda_0(v_0)$. We consider first the formula (5.49) and impose in addition to (H6) the hypothesis

$$(g'(x)x)' \geq 0, \quad x \in (0, \infty). \tag{H7}$$

The latter is the expression of the Baker-Ericksen inequality for stored energies of class (H0) and is implied by rank-one convexity (see Proposition 7.2). Using (5.49) in conjunction to (5.30)₁, (5.33), (H6) and (H7) we obtain

$$\Lambda_0(v_0) > \chi^{-1}(h'(v_0)). \tag{5.50}$$

Next we consider the representation formula (5.47) and impose the hypothesis

$$(g''(x)x)' \geq 0, \quad x \in (0, \infty). \tag{H8}$$

Recall that D_0 is defined in (5.41) and set $a = \dot{\psi}_0$, $b = \frac{\psi_0}{\xi}$. Using (H8), and the facts $a < b$ and $g''' \leq 0$, we obtain

$$\begin{aligned} D_0 &= -b^{2-d}a(ab^{-1} - 1)g''(a) + b^{2-d}(g'(a) - g'(b)) \\ &= b^{1-d}(a - b) \left(-ag''(a) + b \frac{g'(a) - g'(b)}{a - b} \right) \\ &\leq b^{1-d}(b - a)(ag''(a) - bg''(b)) \\ &\leq 0. \end{aligned}$$

In turn, (5.47), (5.40) and (H1) imply the lower bound

$$\Lambda_0(v_0) > \sqrt[d]{v_0}. \tag{5.51}$$

5.4 Convergence to the Limiting Solution

In this section we study the convergence of solutions $(\psi, \delta) = (\psi, \delta)(\xi; \varphi_0, V(\varphi_0))$ of the system (5.6) to solutions $(\psi_0, \delta_0) = (\psi_0, \delta_0)(\xi; V(0))$ of the system (5.30). Using the notations (5.13)–(5.16), (5.40)–(5.41), we rewrite Eqs. (5.6)₂, (5.30)₂ as follows:

$$\begin{aligned} \delta' &= - \left[\frac{d-1}{\psi} \left(1 - \frac{\varphi_0^2 \xi^2}{Q} \right) \frac{D}{f} \right] (\xi; \varphi_0, V(\varphi_0)), \\ \delta'_0 &= - \left[\frac{d-1}{\psi_0} \frac{D_0}{f_0} \right] (\xi; V(0)). \end{aligned} \tag{5.52}$$

Recall also that the initial data satisfy

$$\begin{aligned} \psi(0; \varphi_0, V(\varphi_0)) - \psi_0(0; V(0)) &= 0, \\ \delta(0; \varphi_0, V(\varphi_0)) - \delta_0(0; V(0)) &= V(\varphi_0) - V(0). \end{aligned} \tag{5.53}$$

Proposition 5.6 Assume $d \geq 2$, Φ satisfies (H0)–(H5), and $V : [0, \infty) \rightarrow (0, \infty)$ is a continuous function defined by (5.3). Let $(\psi, \delta), (\psi_0, \delta_0)$ be the solutions of (5.6), (5.30), respectively. Let $\tau > 0$ be fixed. Then, there exists $\widehat{\varepsilon}_\tau > 0$ such that for every

$$0 < \varphi_0 < \widehat{\varepsilon}_\tau$$

the interval $[0, \tau] \subset [0, \mathcal{T}(\varphi_0, V(\varphi_0)))$ and

$$\begin{aligned} & (\psi(\xi; \varphi_0, V(\varphi_0)) - \psi_0(\xi; V(0)))^2 + (\delta(\xi; \varphi_0, V(\varphi_0)) - \delta_0(\xi; V(0)))^2 \\ & \leq \widehat{C}((V(\varphi_0) - V(0))^2 + \varphi_0^4), \quad \forall \xi \in [0, \tau], \end{aligned} \tag{5.54}$$

with the constant $\widehat{C} = \widehat{C}(\tau, V(0)) > 0$ and independent of φ_0 . As a consequence,

$$(\psi, \delta)(\xi; \varphi_0, V(\varphi_0)) \rightarrow (\psi_0, \delta_0)(\xi; V(0)) \quad \text{as } \varphi_0 \rightarrow 0, \tag{5.55}$$

uniformly on $[0, \tau]$.

Proof Fix $\tau > 0$. In view of continuity of V , there exists $\tilde{\varepsilon} > 0$ such that

$$\frac{1}{2}V(0) < V(\varphi_0) < 2V(0), \quad \forall \varphi_0 \in (0, \tilde{\varepsilon}). \tag{5.56}$$

Set $\widehat{\varepsilon}_\tau := \min(\varepsilon_\tau, \tilde{\varepsilon})$, with ε_τ defined in (5.18). Then, Lemma 5.4, (5.35)₁, (5.36), (5.56) imply

$$[0, \tau] \subset [0, \mathcal{T}(\varphi_0, V(\varphi_0))) \quad \text{for every } \varphi_0 \in (0, \widehat{\varepsilon}_\tau)$$

and there exist constants $m_1, m_2, m_3 > 0$ that depend on $\tau, V(0)$ and are independent of φ_0 such that

$$\begin{aligned} 0 < m_1 < \delta(\xi; \varphi_0, V(\varphi_0)), \delta_0(\xi; V(0)) < m_2, \\ 1 \leq \psi(\xi; \varphi_0, V(\varphi_0)), \psi_0(\xi; V(0)) < m_3 \end{aligned} \tag{5.57}$$

for all $\varphi_0 \in (0, \widehat{\varepsilon}_\tau), \xi \in [0, \tau]$.

Next, we fix $\varphi_0 \in (0, \widehat{\varepsilon}_\tau)$ and derive an energy identity that monitors the distance between the solutions (ψ, δ) and (ψ_0, δ_0) of the systems (5.6), (5.30), respectively. First, we subtract (5.30)₁ from (5.6)₁ to get the identity

$$\begin{aligned} \psi' - \psi'_0 &= \delta \left(\frac{\xi}{\psi} \right)^{d-1} - \delta_0 \left(\frac{\xi}{\psi_0} \right)^{d-1} \\ &= (\delta - \delta_0) \left(\frac{\xi}{\psi} \right)^{d-1} + (\psi_0 - \psi) \frac{\delta_0}{\psi_0} \left(\frac{\xi}{\psi} \right)^{d-1} \sum_{i=0}^{d-2} \left(\frac{\psi}{\psi_0} \right)^i. \end{aligned} \tag{5.58}$$

Subtracting the two equations in (5.52) we get

$$\frac{1}{d-1}(\delta' - \delta'_0) = \frac{(D_0 - D)}{\psi_0 f_0} + \frac{(\psi - \psi_0)}{\psi \psi_0 f_0} D + \frac{(f - f_0)}{\psi f f_0} D + \frac{\varphi_0^2 \xi^2}{\widehat{Q} \psi f} D. \tag{5.59}$$

Multiplying (5.58) by $(\psi - \psi_0)$ and (5.59) by $(\delta - \delta_0)(d - 1)$ and adding the results we obtain

$$\begin{aligned} & \frac{d}{d\xi} \left(\frac{1}{2}(\psi - \psi_0)^2 + \frac{1}{2}(\delta - \delta_0)^2 \right) \\ &= (\psi - \psi_0)(\delta - \delta_0) \left(\frac{\xi}{\psi} \right)^{d-1} - (\psi - \psi_0)^2 \frac{\delta_0}{\psi_0} \left(\frac{\xi}{\psi} \right)^{d-1} \sum_{i=0}^{d-2} \left(\frac{\psi}{\psi_0} \right)^i \\ & \times (d - 1) \left[-(D - D_0)(\delta - \delta_0) \frac{1}{\psi_0 f_0} + (\psi - \psi_0)(\delta - \delta_0) \frac{D}{\psi \psi_0 f_0} \right. \\ & \left. + (f - f_0)(\delta - \delta_0) \frac{D}{\psi f f_0} + (\delta - \delta_0) \frac{D\varphi_0^2 \xi^2}{\widehat{Q}\psi f} \right], \quad \xi \in (0, \tau]. \end{aligned} \tag{5.60}$$

We now estimate the right-hand side of (5.60). By (5.57) we get for $\xi \in (0, \tau]$

$$0 < \left(\frac{\xi}{\psi} \right)^{d-1} \leq \tau^{d-1}, \quad \left| \frac{\delta_0}{\psi_0} \left(\frac{\xi}{\psi} \right)^{d-1} \sum_{i=0}^{d-2} \left(\frac{\psi}{\psi_0} \right)^i \right| \leq \tau^{d-1} m_2 \sum_{i=0}^{d-2} m_3^i. \tag{5.61}$$

Similarly, by (H2), (5.27), (5.13), (5.40), and (5.57), we obtain

$$\left| \frac{1}{\psi_0 f_0} \right| + \left| \frac{1}{\psi \psi_0 f_0} \right| + \left| \frac{1}{\psi f f_0} \right| + \left| \frac{\xi^2}{\widehat{Q}\psi f} \right| \leq \frac{\max(1, h''(m_2))}{(h''(m_2))^2} \left(3 + \frac{2\tau^2}{\nu^2} \right). \tag{5.62}$$

Also, using (H2), (5.16), (5.57) and the inequalities (5.23)₂, (5.25), we obtain

$$|D(\xi; \varphi_0, V(\varphi_0))| \leq \tau^{2d-3} m_2 (\tau^2 \varphi_0^2 + g''(0)) + 2\widehat{\gamma}(1 + \tau^{d-2}). \tag{5.63}$$

Combining (5.60)–(5.63), and using Young’s inequality, we get for $\xi \in (0, \tau]$

$$\begin{aligned} & \frac{d}{d\xi} ((\psi - \psi_0)^2 + (\delta - \delta_0)^2) \\ & \leq C((\psi - \psi_0)^2 + (\delta - \delta_0)^2 + (f - f_0)^2 + (D - D_0)^2 + \varphi_0^4), \end{aligned} \tag{5.64}$$

with $C = C(\tau, V(0)) > 0$ independent of φ_0 .

By (5.57), (5.58) and (5.61) for all $\xi \in (0, \tau]$

$$\begin{aligned} & |f(\xi; \varphi_0, V(\varphi_0)) - f_0(\xi; V(0))| \\ & \leq |h''(\delta) - h''(\delta_0)| + g'' \left(\delta_0 \left(\frac{\xi}{\psi_0} \right)^{d-1} \right) \left| \left(\frac{\xi}{\psi} \right)^{2d-2} - \left(\frac{\xi}{\psi_0} \right)^{2d-2} \right| \\ & \quad + \left(\frac{\xi}{\psi} \right)^{2d-2} \left| g'' \left(\delta \left(\frac{\xi}{\psi} \right)^{d-1} \right) - g'' \left(\delta_0 \left(\frac{\xi}{\psi_0} \right)^{d-1} \right) \right| \\ & \leq |\delta - \delta_0| \sup_{x \in [m_1, m_2]} |h'''(x)| + \tau^{2d-2} |\psi_0 - \psi| g''(0) \sum_{i=0}^{2d-3} m_3^i \\ & \quad + \tau^{3d-3} \left(|\delta - \delta_0| + |\psi_0 - \psi| m_2 \sum_{i=0}^{d-2} m_3^i \right) \sup_{x \in (0, m_2 \tau^{d-1})} |g'''(x)|. \end{aligned} \tag{5.65}$$

Next, by (5.16) and (5.41)

$$\begin{aligned}
 & |D(\xi; \varphi_0, V(\varphi_0)) - D_0(\xi; V(0))| \\
 & \leq \left\{ \left| \left(\frac{\xi}{\psi} \right)^{2d-3} \delta \left(\delta \left(\frac{\xi}{\psi} \right)^d - 1 \right) \xi^2 \varphi_0^2 \right| \right. \\
 & \quad + \left| \left(\frac{\xi}{\psi_0} \right)^{2d-3} - \left(\frac{\xi}{\psi} \right)^{2d-3} \right| \left| \delta_0 \left(\delta_0 \left(\frac{\xi}{\psi_0} \right)^d - 1 \right) g'' \left(\delta_0 \left(\frac{\xi}{\psi_0} \right)^{d-1} \right) \right| \\
 & \quad + |\delta_0 - \delta| \left| \left(\frac{\xi}{\psi} \right)^{2d-3} \left(\delta_0 \left(\frac{\xi}{\psi_0} \right)^d - 1 \right) g'' \left(\delta_0 \left(\frac{\xi}{\psi_0} \right)^{d-1} \right) \right| \\
 & \quad + \left| \delta_0 \left(\frac{\xi}{\psi_0} \right)^d - \delta \left(\frac{\xi}{\psi} \right)^d \right| \left| \delta \left(\frac{\xi}{\psi} \right)^{2d-3} g'' \left(\delta_0 \left(\frac{\xi}{\psi_0} \right)^{d-1} \right) \right| \\
 & \quad + \left| g'' \left(\delta_0 \left(\frac{\xi}{\psi_0} \right)^{d-1} \right) - g'' \left(\delta \left(\frac{\xi}{\psi} \right)^{d-1} \right) \right| \left| \left(\delta \left(\frac{\xi}{\psi} \right)^d - 1 \right) \delta \left(\frac{\xi}{\psi} \right)^{2d-3} \right| \Big\} \\
 & \quad + \left| \left(\frac{\xi}{\psi} \right)^{d-2} g' \left(\delta \left(\frac{\xi}{\psi} \right)^{d-1} \right) - \left(\frac{\xi}{\psi_0} \right)^{d-2} g' \left(\delta_0 \left(\frac{\xi}{\psi_0} \right)^{d-1} \right) \right| \\
 & \quad + \left| \left(\frac{\xi}{\psi} \right)^{d-2} g' \left(\frac{\psi}{\xi} \right) - \left(\frac{\xi}{\psi_0} \right)^{d-2} g' \left(\frac{\psi_0}{\xi} \right) \right| =: J_1 + J_2 + J_3. \tag{5.66}
 \end{aligned}$$

Using (H2), (5.23)₂, (5.57), (5.58), and (5.61) we obtain

$$\begin{aligned}
 J_1 & \leq \tau^{2d-1} m_2 \varphi_0^2 + \tau^{2d-3} |\psi_0 - \psi| g''(0) m_2 \sum_{i=0}^{2d-4} m_3^i + \tau^{2d-3} g''(0) |\delta_0 - \delta| \\
 & \quad + m_2 \tau^{3d-3} g''(0) \left(|\delta - \delta_0| + |\psi_0 - \psi| m_2 \sum_{i=0}^{d-1} m_3^i \right) \\
 & \quad + m_2 \tau^{3d-4} \left(|\delta - \delta_0| + |\psi_0 - \psi| m_2 \sum_{i=0}^{d-2} m_3^i \right) \sup_{x \in [0, m_2 \tau^{d-1}]} |g'''(x)|. \tag{5.67}
 \end{aligned}$$

To estimate terms J_2, J_3 we consider two separate cases for the constant d .

Case $d = 2$. First, using (H2), (5.58), (5.61), we obtain

$$J_2 \leq g''(0) \tau^{d-1} \left(|\delta - \delta_0| + |\psi_0 - \psi| m_2 \sum_{i=0}^{d-2} m_3^i \right). \tag{5.68}$$

Then by (H1), (H2), and (H3) we obtain for $x \in [0, \infty)$

$$0 \leq g''(x)x \leq \int_0^1 g''(sx)x ds = g'(x) - g'(0) \leq \gamma - g'(0). \tag{5.69}$$

Then, using (H2), (5.69), and the fact that $\psi, \psi_0 \geq 1$, we obtain

$$J_3 = \left| g' \left(\frac{\psi}{\xi} \right) - g' \left(\frac{\psi_0}{\xi} \right) \right| \leq g'' \left(\frac{1}{\xi} \right) \left| \frac{\psi}{\xi} - \frac{\psi_0}{\xi} \right| \leq (\gamma - g'(0)) |\psi - \psi_0|. \tag{5.70}$$

Case $d \geq 3$. In that case, using (5.57) and (5.35), we get

$$\begin{aligned}
 J_2 \leq & g''(0)\tau^{2d-3} \left(|\delta - \delta_0| + |\psi_0 - \psi| m_2 \sum_{i=0}^{d-2} m_3^i \right) \\
 & + \tau^{d-2} |\psi_0 - \psi| \sum_{i=0}^{d-3} m_3^i \sup_{x \in [0, A_0]} |g'(x)|
 \end{aligned}
 \tag{5.71}$$

and, similarly, using (H2) and the bound (5.25), we obtain

$$\begin{aligned}
 J_3 \leq & \left| \left(\frac{\xi}{\psi} \right)^{d-2} - \left(\frac{\xi}{\psi_0} \right)^{d-2} \right| \left| g' \left(\frac{\psi}{\xi} \right) \right| + \left(\frac{\xi}{\psi_0} \right)^{d-2} \left| g' \left(\frac{\psi}{\xi} \right) - g' \left(\frac{\psi_0}{\xi} \right) \right| \\
 \leq & \left(|\psi_0 - \psi| \sum_{i=0}^{d-3} m_3^i \right) \left| \left(\frac{\xi}{\psi} \right)^{d-2} g' \left(\frac{\psi}{\xi} \right) \right| + \xi^{d-2} g''(0) \left| \frac{\psi}{\xi} - \frac{\psi_0}{\xi} \right| \\
 \leq & \left(|\psi_0 - \psi| \sum_{i=0}^{d-3} m_3^i \right) 2\widehat{\gamma}(1 + \tau^{d-2}) + \tau^{d-3} g''(0) |\psi - \psi_0|.
 \end{aligned}
 \tag{5.72}$$

Combining the estimates (5.64)–(5.72) we obtain

$$\frac{d}{d\xi} ((\psi - \psi_0)^2 + (\delta - \delta_0)^2) \leq C((\psi - \psi_0)^2 + (\delta - \delta_0)^2 + \varphi_0^4), \quad \xi \in (0, \tau], \tag{5.73}$$

with $C = C(\tau, v_0) > 0$ independent of φ_0 . Then Gronwall’s lemma and (5.53) yield (5.54) and (5.55). □

5.5 The Critical Stretching for Dynamic Bifurcation

By Proposition 5.5 for each $v_0 > 0$ there exists a unique $\Lambda_0 = \Lambda_0(v_0)$ such that

$$0 < \Lambda_0(v_0) = \lim_{\xi \rightarrow \infty} \frac{\psi_0(\xi; v_0)}{\xi} = \lim_{\xi \rightarrow \infty} \psi'_0(\xi; v_0), \tag{5.74}$$

where $(\psi_0, \delta_0)(\xi; v_0)$ is a global solution of (5.30). In this section, we will show that $\Lambda_0(V(0))$, with V defined by (5.3), is the critical stretching for *dynamic bifurcation* from the uniformly deformed state for the system (3.14), (3.20).

Theorem 5.7 *Assume $d \geq 2$, Φ satisfies (H0)–(H5), and V is defined by (5.3). Let (φ, v) be as in Theorem 3.3, and let $\sigma, \Lambda, \Lambda_0$ be defined by (5.1), (5.2), and (5.74), respectively. Then,*

(i)

$$\lim_{\varphi_0 \rightarrow 0^+} \Lambda(\varphi_0, V(\varphi_0)) = \Lambda_0(V(0)). \tag{5.75}$$

(ii) *The strength of the shock and its speed satisfy*

$$\lim_{\varphi_0 \rightarrow 0^+} \left[\frac{\varphi}{s} - \dot{\varphi} \right] (\sigma(\varphi_0, V(\varphi_0)); \varphi_0, V(\varphi_0)) = 0, \tag{5.76}$$

$$\lim_{\varphi_0 \rightarrow 0_+} \sigma(\varphi_0, V(\varphi_0)) = \sqrt{\Phi_{11}(\Lambda_0(V(0)), \Lambda_0(V(0)))} =: \sigma_0 > 0. \tag{5.77}$$

(iii) *The solutions of (3.14) satisfy*

$$\begin{aligned} \lim_{\varphi_0 \rightarrow 0_+} \varphi(s; \varphi_0, V(\varphi_0)) &= \Lambda_0(V(0))s, \quad 0 \leq s < \sigma_0, \\ \lim_{\varphi_0 \rightarrow 0_+} v(s; \varphi_0, V(\varphi_0)) &= \begin{cases} V(0), & s = 0, \\ [\Lambda_0(V(0))]^d, & 0 < s < \sigma_0. \end{cases} \end{aligned} \tag{5.78}$$

Proof We recall that $(\varphi, v)(s; \varphi_0, V(\varphi_0))$ are defined for $s \in [0, \sigma(\varphi_0, V(\varphi_0))]$ and solve the initial value problem (3.14) with $v_0 = V(\varphi_0)$. At the point $\sigma := \sigma(\varphi_0, V(\varphi_0))$ there is a shock (or sonic singularity). If we have a shock at σ , then (3.30), (5.1), (3.23), (H2) and (H5) imply

$$\sigma(\varphi_0, V(\varphi_0)) = \sqrt{\frac{\Phi_1(\frac{\varphi}{s}, \frac{\varphi}{s}) - \Phi_1(\frac{\varphi}{s}, \frac{\varphi}{s})}{\varphi - \frac{\varphi}{s}} \Big|_{s=\sigma}} \geq \sqrt{\Phi_{11}\left(\frac{\varphi}{s}, \frac{\varphi}{s}\right) \Big|_{s=\sigma}} \geq \nu, \quad \forall \varphi_0 > 0. \tag{5.79}$$

If σ is a sonic singularity then the same conclusion follows from Theorem 4.1(ii), in conjunction with (3.12) and (3.25).

By (5.5) the rescaled functions $(\psi, \delta)(\xi; \varphi_0, V(\varphi_0))$ are well-defined for $\xi \in [0, \xi^*(\varphi_0)]$, where

$$\xi^*(\varphi_0) := \frac{\sigma(\varphi_0, V(\varphi_0))}{\varphi_0}, \tag{5.80}$$

and satisfy the initial value problem (5.6). Note that by (5.2) and (5.5)

$$\Lambda(\varphi_0, V(\varphi_0)) = \left[\frac{\varphi}{s} \right] (\sigma(\varphi_0); \varphi_0, V(\varphi_0)) = \left[\frac{\psi}{\xi} \right] (\xi^*(\varphi_0), \varphi_0, V(\varphi_0)). \tag{5.81}$$

Consider the function $\psi = \psi(\xi; \varphi_0, V(\varphi_0))$ and recall that $\psi - \xi\psi' > 0$ and $\frac{d}{d\xi}(\psi - \xi\psi') < 0$. Hence

$$0 < \frac{\psi}{\xi} - \psi' < \frac{1}{\xi}, \quad 0 < \xi \leq \xi^*(\varphi_0), \tag{5.82}$$

and therefore

$$0 < -\frac{d}{d\xi} \left(\frac{\psi(\xi; \varphi_0, V(\varphi_0))}{\xi} \right) = \frac{1}{\xi} \left(\frac{\psi}{\xi} - \psi' \right) < \frac{1}{\xi^2}, \quad 0 < \xi \leq \xi^*(\varphi_0). \tag{5.83}$$

Then (5.81) and (5.83) imply

$$0 < \frac{\psi(\tau; \varphi_0, V(\varphi_0))}{\tau} - \Lambda(\varphi_0, V(\varphi_0)) = - \int_{\tau}^{\xi^*(\varphi_0)} \frac{d}{d\xi} \left(\frac{\psi}{\xi} \right) d\xi < \frac{1}{\tau}, \quad 0 < \tau \leq \xi^*(\varphi_0). \tag{5.84}$$

Next, we consider the functions $(\psi_0, \delta_0)(\xi; V(0))$ defined on $[0, \infty)$ and satisfying (5.30) with $v_0 = V(0)$. A similar argument for ψ_0 shows that

$$0 < \frac{\psi_0(\tau; V(0))}{\tau} - \Lambda_0(V(0)) < \int_{\tau}^{\infty} \frac{1}{\xi^2} d\xi = \frac{1}{\tau}, \quad 0 < \tau < \infty. \tag{5.85}$$

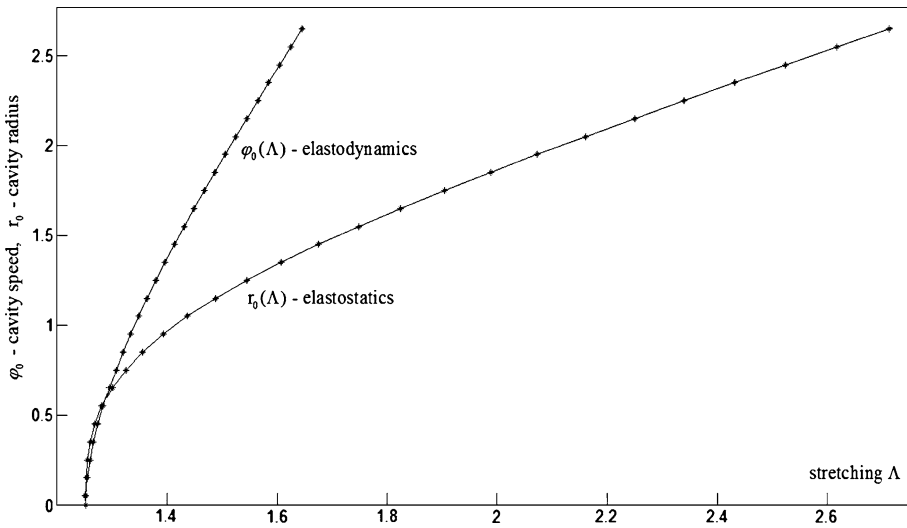


Fig. 2 Bifurcation curves for statics and dynamics: $g(x) = \frac{1}{2}x^2, h(x) = (x - 1) \ln(x), T_{rad}(0) = 0$

We proceed to show (5.75). Fix $\varepsilon > 0$ and select $\tau = (3/\varepsilon)$ and $\bar{\alpha} = \frac{\varepsilon v}{3}$. If we restrict $\varphi_0 \in [0, \bar{\alpha}]$ then (5.79), (5.80) imply that the interval

$$[0, \tau] \subset [0, \xi^*(\varphi_0)] \subset [0, \mathcal{T}(\varphi_0, V(\varphi_0))].$$

Proposition 5.6 then implies, by restricting α further (if necessary), that for $\varphi_0 \in [0, \alpha]$

$$\left| \frac{\psi(\tau; \varphi_0, V(\varphi_0))}{\tau} - \frac{\psi_0(\tau; V(0))}{\tau} \right| < \frac{\varepsilon}{3}. \tag{5.86}$$

Then combining (5.86), (5.84) and (5.85) we arrive at the desired

$$|\Lambda(\varphi_0, V(\varphi_0)) - \Lambda_0(V(0))| < \varepsilon.$$

We next use the scaling transformation (5.5) to re-express the inequalities (5.84) and (5.82) into the forms

$$0 < \frac{\varphi(s)}{s} - \Lambda(\varphi_0, V(\varphi_0)) < \frac{\varphi_0}{s}, \quad 0 < s \leq \sigma(\varphi_0, V(\varphi_0)), \tag{5.87}$$

$$0 < \frac{\varphi(s)}{s} - \dot{\varphi}(s) < \frac{\varphi_0}{s}, \quad 0 < s \leq \sigma(\varphi_0, V(\varphi_0)). \tag{5.88}$$

From these, in conjunction with (5.75), we deduce (5.76) and (5.78). Finally, (5.77) follows by passing to the limit $\varphi_0 \rightarrow 0_+$ in (5.79) using (5.87) and (5.88). \square

Remark 5.8 Given a relation $T_{rad}(0; \varphi_0, v_0) = G(\varphi_0)$, with G continuous at $\varphi_0 = 0$, Theorem 5.7 states that the critical point for a cavity with content is given by

$$\Lambda_0(h^{-1}(G(0))) = \lim_{\varphi_0 \rightarrow 0_+} \Lambda(\varphi_0, h^{-1}(G(\varphi_0))). \tag{5.89}$$

For a stress-free cavity ($G(\varphi_0) \equiv 0$) the result (5.89) means that

$$\lim_{\varphi_0 \rightarrow 0_+} \Lambda(\varphi_0, H) = \Lambda_0(H) = \lambda_{cr},$$

where λ_{cr} is the *critical stretching* associated with cavitating solutions for equilibrium elasticity analyzed in [2, Sect. 7.5]; see also Sect. 5.3.2. Thus, the critical values of the dynamic and equilibrium bifurcation diagrams (in the stress-free case) coincide. In Fig. 2 the bifurcation diagrams for the dynamic and the equilibrium radial elasticity are compared numerically for stress-free cavities.

6 Conclusions, Open Problems

As already mentioned, this article complements [18, 19] and completes a theory of dynamically cavitating weak solutions for (1.2), the equations of dynamic, radial elasticity for isotropic materials. The solutions are self-similar and satisfy (1.4), (1.5) and (1.6). The emerging solution is a regular weak solution of the equations of radial elasticity, which consists of an opening cavity at the center, followed by a smoothly varying part, and then by a precursor to the cavity shock, that connects the smooth part of the solution to a uniform deformation at the far field.

There are two techniques for constructing the wave pattern that forms the cavitating solution. According to one approach, introduced in [18] and followed here, one starts from the cavity center and proceeds to connect through a shock to the uniformly deformed state. An alternative, followed in [19], is to start from the outer part of the wave, the uniform deformation at the far field, and construct a wave pattern followed by a shock and the smooth part of the cavitating solution, using a shooting argument to guarantee that the radial Cauchy stress achieves the value zero (and thus a cavity) at the center. Both techniques have their advantages, however the first approach provides a way of handling any cavitating solution, and it serves as a starting point to carry out the analysis of the dynamic bifurcation curve. It turns out that there is a critical stretching λ_{cr} for obtaining cavitating solutions and it coincides with the critical stretching predicted by the associated equilibrium elasticity problem.

The resulting solution offers a striking example of non uniqueness for the equations of radial isotropic elastodynamics, because, as noted in [18, Theorem 7.2], the total mechanical energy of the solution with the cavity is less than the total energy of the homogeneous deformation. In the parlance of conservation law theory, this provides an example of non-uniqueness of entropy weak solutions (for polyconvex energies) due to point-singularities at the cavity. As opening a cavity decreases the energy, this provides an autocatalytic mechanism for material failure. This paradox was resolved in [8], where the question is raised whether remaining at the level of weak solutions is sufficient for describing singular phenomena like cavitation or shear bands, that lie at the limits (perhaps even outside) of continuum modeling. The notion of singular limiting induced by continuum (*slic*)-solution is introduced, according to which a singular solution is a *slic*-solution if it can be realized as limit of spatial averagings of the singular weak solution. This definition is tested for the cavitating solution, and it turns out that local spatial averaging produces a surface energy cost at the opening cavity that renders the uniform deformation the energetically preferred solution (see [8]).

An important open problem is thus to come up with a solution concept that accounts for surface energies on singular objects. The calculus of variations literature has done steps in this direction (e.g., [9, 17] and references therein) but there is currently no available notion

of solution that accounts for surface energies on singular objects and is applicable at the dynamic level.

Another open problem is the following: Despite the existing constructions of cavitating weak solution obtained in [18, 19] and here, there is no simple understanding of the mechanism of loss of stability of the uniform deformation that leads to cavitation, or how such a criterion relates to the critical stretching computed here. This is an important research direction for the dynamic problem that needs to be understood.

Appendix

7.1 Gradients of Radial Functions

Consider a radial function $y : \mathbb{R}^d \rightarrow \mathbb{R}^d$ of the form

$$y(x) = w(R) \frac{x}{R}, \quad R := |x| \text{ with } w(R) : (0, \infty) \rightarrow \mathbb{R}. \tag{7.1}$$

Theorem 7.1 (J. Ball [2]) *Let $d > 1$, let $1 \leq p < \infty$ and y be given by (7.1). Then:*

(i) $y \in L^p_{loc}(\mathbb{R}^d)$ if and only if

$$\int_0^\rho R^{d-1} |w(R)|^p dR < \infty \quad \text{for all } \rho \in (0, \infty).$$

(ii) $y \in W^{1,p}_{loc}(\mathbb{R}^d)$ if and only if $w(R)$ is absolutely continuous on $(0, \infty)$ and

$$\int_0^\rho R^{d-1} \left(|w_R|^p + \left| \frac{w}{R} \right|^p \right) dR < \infty \quad \text{for all } \rho \in (0, \infty). \tag{7.2}$$

(iii) If (7.2) holds for (say) $\rho = 1$ then

$$|w(R)|^p R^{d-p} \rightarrow 0 \quad \text{as } R \rightarrow 0. \tag{7.3}$$

(iv) If $y \in W^{1,1}_{loc}(\mathbb{R}^d)$ then

$$\nabla y(x) = w_R \frac{x \otimes x}{R^2} + \frac{w}{R} \left(\mathbf{I} - \frac{x \otimes x}{R^2} \right) \quad \text{in } \mathcal{D}'(\mathbb{R}^d) \text{ and a.e. } x \in \mathbb{R}^d. \tag{7.4}$$

For the proofs of (i), (ii) and (iv) we refer to [2].

The proof of (iii) for $p = 1$ goes as follows: Using the identity

$$w(R)R^{d-2} = w(\rho)\rho^{d-2} + \int_\rho^R \left(w_s(s)s^{d-2} + (d-2)\frac{w(s)}{s}s^{d-2} \right) ds, \quad \text{for } 0 < \rho < R,$$

we integrate over $(0, R)$ and use Fubini to obtain

$$w(R)R^{d-1} = \int_0^R \left(w_\rho + (d-1)\frac{w}{\rho} \right) \rho^{d-1} d\rho. \tag{7.5}$$

This identity holds for smooth functions; then using a density argument one establishes the identity for functions $w \in W^{1,1}_{loc}(\mathbb{R}^d)$.

For $p = 1$, since the integral in (7.2) is finite, we have

$$F(R) := \int_0^R \left(w_\rho + (d - 1) \frac{w}{\rho} \right) d\rho \rightarrow 0 \quad \text{as } R \rightarrow 0$$

(in fact the function $F(R)$ is absolutely continuous as function of R). Hence (iii) follows from (7.5) for $p = 1$. The case $p > 1$ is done by a similar argument.

In order to compute the distributional derivative of ∇y in (iv), one may follow the usual process of deleting a ball of small radius $\varepsilon > 0$ around the origin, using the formula of integration by parts and passing to the limit $\varepsilon \rightarrow 0$. Then, the contribution from the surface of the ball will vanish precisely because of (7.3) and thus no delta mass appears in the formula (7.4) for dimensions $d \geq 2$.

7.2 Stored Energies

We collect here certain properties of the stored energies that are used throughout this study. As already mentioned, frame indifference and isotropy are equivalent to expressing the stored energy as

$$W(F) = \Phi(v_1, v_2, \dots, v_d),$$

where $\Phi : \mathbb{R}_+^d \rightarrow \mathbb{R}$ is a symmetric function of its arguments and v_1, \dots, v_d are the eigenvalues of the positive square root $(F^T F)^{\frac{1}{2}}$, the so called principal stretches [1, 22].

The stored energy $W(F)$ is said to be *rank-1 convex* if

$$W(\tau F + (1 - \tau)G) \leq \tau W(F) + (1 - \tau)W(G), \tag{7.6}$$

for $0 < \tau < 1$ and for $F, G \in M_+^{d \times d}$ such that $F - G = \xi \otimes v$ for some nonzero $\xi, v \in \mathbb{R}^d$. If the inequality in (7.6) is strict, then W is called *strictly rank-1 convex*.

It is easy to check that for $W \in C^2(M_+^{d \times d})$ rank-1 convexity is equivalent to the *Legendre-Hadamard condition*, that is

$$\frac{\partial^2 W(F)}{\partial F_{i\alpha} \partial F_{j\beta}} \xi^i v_\alpha \xi^j v_\beta \geq 0, \quad \forall F \in M_+^{d \times d} \text{ and } \forall \xi, v \in \mathbb{R}^d - \{0\}.$$

For isotropic rank-1 convex functions, the stored energy Φ must satisfy certain monotonicity properties:

Proposition 7.2 (J. Ball [2]) *Let $W \in C^1(M_+^{d \times d})$ be strictly rank-1 convex and isotropic. Then:*

- (i) $\frac{\partial \Phi}{\partial v_i}(v_1, \dots, v_d)$ is a strictly increasing function of v_i when $v_j, j \neq i$ are kept fixed. If in addition $W \in C^2(M_+^{d \times d})$ then $\frac{\partial^2 \Phi}{\partial v_i^2}(v_1, \dots, v_d) > 0$.
- (ii) The Baker-Ericksen inequalities hold, that is

$$\left[\frac{v_i \frac{\partial \Phi}{\partial v_i} - v_j \frac{\partial \Phi}{\partial v_j}}{v_i - v_j} \right] > 0 \quad \text{for } i \neq j, v_i \neq v_j. \tag{7.7}$$

Throughout this study we work with stored energies of the special form

$$\Phi(v_1, v_2, \dots, v_d) = \sum_{i=1}^d g(v_i) + h(v_1 v_2 \dots v_d), \tag{H0}$$

where the functions $g(x) \in C^3[0, \infty)$ and $h(x) \in C^3(0, \infty)$. One easily computes their derivatives,

$$\begin{aligned} \Phi_{11} &= g''(v_1) + (v_2 \dots v_d)^2 h''(v), \\ \Phi_{12} &= (v_3 \dots v_d) h'(v) + v_1 v_2 (v_3 \dots v_d)^2 h''(v), \\ P &= \Phi_{12} + \frac{\Phi_1 - \Phi_2}{v_1 - v_2} = \frac{g'(v_1) - g'(v_2)}{v_1 - v_2} + (v_3 \dots v_d) v h''(v), \\ \Phi_{111} &= g'''(v_1) + (v_2 \dots v_d)^3 h'''(v), \\ \Phi_{112} &= v_2 (v_3 \dots v_d)^2 [2h''(v) + v h'''(v)], \end{aligned}$$

where $v = v_1 v_2 \dots v_d$.

Due to the form of the principal stretches for radial motions (see (7.4)), in the problem of cavitation it is often needed to work for (v_1, \dots, v_d) taking values of the form (a, b, \dots, b) or on the diagonal (b, b, \dots, b) . The symmetry of Φ entails certain properties on the diagonals:

$$\frac{\partial \Phi}{\partial v_i}(a, b, \dots, b) = \frac{\partial \Phi}{\partial v_j}(a, b, \dots, b) \quad \text{for } i, j = 2, \dots, d, \quad i \neq j, \quad \forall a, b > 0, \tag{7.8}$$

$$\frac{\partial \Phi}{\partial v_1}(b, b, \dots, b) = \frac{\partial \Phi}{\partial v_j}(b, b, \dots, b) \quad \text{for } j \neq 1, \quad \forall b > 0. \tag{7.9}$$

When working with stored energies computed along the sets (a, b, \dots, b) we will often use the short hand notation

$$\begin{aligned} \Phi_1(a, b) &\equiv \frac{\partial \Phi}{\partial v_1}(a, b, \dots, b), & \Phi_2(a, b) &\equiv \frac{\partial \Phi}{\partial v_j}(a, b, \dots, b), \quad j = 2, \dots, d, \\ \Phi_{11}(a, b) &\equiv \frac{\partial^2 \Phi}{\partial v_1^2}(a, b, \dots, b), & \Phi_{12}(a, b) &\equiv \frac{\partial^2 \Phi}{\partial v_1 \partial v_j}(a, b, \dots, b), \quad j = 2, \dots, d, \end{aligned} \tag{7.10}$$

and so on.

The quantity

$$P(a, b) := \begin{cases} \Phi_{12}(a, b, \dots, b) + \frac{(\Phi_1 - \Phi_2)(a, b, \dots, b)}{a - b}, & a < b, \\ \Phi_{11}(b, b, \dots, b), & a = b, \end{cases}$$

appears in the defining differential equation (3.3). Using (7.10) one checks that

$$\lim_{\substack{a \rightarrow \lambda^- \\ b \rightarrow \lambda^+}} P(a, b) = \lim_{\substack{a \rightarrow \lambda^- \\ b \rightarrow \lambda^+}} \Phi_{11}(a, b, \dots, b) = \Phi_{11}(\lambda, \dots, \lambda) \tag{7.11}$$

and thus $P(a, b)$ is continuous up to the diagonal on the set $\{(a, b) : 0 < a \leq b\}$. Furthermore, using (7.8), (7.9) and Taylor expansions around the diagonal we easily see that

$$\lim_{\substack{a \rightarrow \lambda^- \\ b \rightarrow \lambda^+}} \frac{P(a, b) - \Phi_{11}(a, b)}{a - b} = \frac{1}{2} (\Phi_{112}(\lambda, \lambda) - \Phi_{111}(\lambda, \lambda)). \tag{7.12}$$

We list some formulas based on (H0) that are used in the text:

$$\Phi_{11}(a, b) = g''(a) + b^{2d-2} h''(ab^{d-1}),$$

$$\begin{aligned} \Phi_{12}(a, b) &= b^{d-2}h'(ab^{d-1}) + ab^{2d-3}h''(ab^{d-1}), \\ P(a, b) &= \Phi_{12} + \frac{\Phi_1 - \Phi_2}{a - b} = \frac{g'(a) - g'(b)}{a - b} + ab^{2d-3}h''(ab^{d-1}), \\ Q(a, b, s) &= s^2 - \Phi_{11} = s^2 - [g''(a) + b^{2d-2}h''(ab^{d-1})], \\ \Phi_{111}(a, b) &= g'''(a) + b^{3d-3}h'''(ab^{d-1}), \\ \Phi_{112}(a, b) &= b^{2d-3}[2h''(ab^{d-1}) + ab^{d-1}h'''(ab^{d-1})], \\ R(a, b, s) &= \begin{cases} \frac{\Phi_1(a,b) - \Phi_1(b,b)}{a-b} - s^2, & a < b, \\ \Phi_{11}(b, b) - s^2, & a = b, \end{cases} \\ &= \begin{cases} \left(\frac{g'(a) - g'(b)}{a-b}\right) + b^{2d-2}\left(\frac{h'(ab^{d-1}) - h'(b^d)}{ab^{d-1} - b^d}\right) - s^2, & a < b, \\ g''(b) + b^{2d-2}h''(b^d) - s^2, & a = b. \end{cases} \end{aligned}$$

List of Hypotheses For the reader’s convenience, we collect the hypotheses used in the analysis of the dynamic bifurcation problem:

$$\Phi(v_1, v_2, \dots, v_d) = \sum_{i=1}^d g(v_i) + h(v_1 v_2 \dots v_d), \quad g \in C^3[0, \infty), h \in C^3(0, \infty), \quad (H0)$$

$$g''(x) > 0, \quad h''(x) > 0, \quad \lim_{x \rightarrow 0} h(x) = \lim_{x \rightarrow \infty} h(x) = +\infty, \quad (H1)$$

$$g'''(x) \leq 0, \quad h'''(x) < 0, \quad (H2)$$

$$\lim_{x \rightarrow \infty} \left(\frac{g'(x)}{x^{d-2}}\right) = \gamma \geq 0, \quad (H3)$$

$$h'(x) \rightarrow -\infty \text{ as } x \rightarrow 0_+, \quad h'(x) \rightarrow +\infty \text{ as } x \rightarrow +\infty, \quad (H4)$$

$$\Phi_{11}(x, x) = g''(x) + x^{2d-2}h''(x^d) \geq v^2 > 0. \quad (H5)$$

(H0)–(H3) and (H4)₂ play a role in the existence of a weak solution with cavity, while, in addition to them, (H4)₁ and (H5) are used in the dynamic bifurcation problem. An example of stored energy that satisfies (H0)–(H5) is:

Case $d \geq 3$: $g(x), h(x)$ in (H0) are selected by

$$g(x) = \sum_{k=1}^K a_k(x + \varepsilon_k)^{\alpha_k} \quad \text{with } 1 < \alpha_k \leq 2, a_k, \varepsilon_k > 0, \quad (7.13)$$

and

$$h(x) = \sum_{m=1}^M b_m x^{\beta_m} + \sum_{n=1}^N c_n x^{-\mu_n} \quad \text{with } 1 < \beta_m \leq 2, \mu_n > 0, b_m, c_n > 0. \quad (7.14)$$

Case $d \geq 2$: $g(x)$ is selected by

$$g(x) = ax + \sum_{k=1}^K \frac{b_k}{(x + \varepsilon_k)^{\alpha_k}} \quad \text{with } \alpha_k, \varepsilon_k, b_k > 0, a \geq 0, \quad (7.15)$$

while $h(x)$ is the same as in (7.14).

We note that $\varepsilon_k, k = 1, \dots, K$, in (7.13) and (7.15), are chosen positive in order to satisfy the requirement that $g \in C^3[0, \infty)$. Thus, $g''(x)$ cannot blow up as $x \rightarrow 0_+$ in view of the requirement that $\varepsilon_k > 0$. This restricts the class of stored energies as compared to the class of Ogden materials [11]; see also [2, p. 593]. Also we note that $h(x)$ defined in (7.14) satisfies $\limsup_{x \rightarrow \infty} h''(x^d)x^{2d-2} > 0$ for $d \geq 2$ and this together with the fact that $g''(x) > 0, x \in [0, \infty)$, gives (H5).

In Sect. 5.3.2, we also used the hypotheses

$$\frac{d}{dx}(h'(x^d) + g'(x)x^{1-d}) > 0, \quad \limsup_{x \rightarrow 0_+}(h'(x)x) < 0, \tag{H6}$$

$$(g'(x)x)' > 0, \tag{H7}$$

$$(g''(x)x)' \geq 0. \tag{H8}$$

These play a very limited role, solely in establishing bounds for the critical stretching λ_{cr} of the equilibrium elasticity critical stretching. Namely, (H6) and (H7) are used for obtaining the bound (5.50), while (H8) is used in the derivation of the lower bound (5.51). For stored energies of class (H0), (H7) expresses the Baker-Ericksen inequality (7.7).

7.3 Numerical Computations

We now briefly discuss the numerical computation used to plot the graphs of $v(s; \varphi_0, H)$ in Fig. 1 and the bifurcation curves in Fig. 2. To obtain the solution $(\varphi, v)(s; \varphi_0, v_0)$ of (3.14) we perform numerical computations employing the original (equivalent) system (3.4). As (3.4) has a geometric singularity at the origin, we initiate the solution using an analytical argument to depart from the singularity at $s = 0$, and once we are off the singularity we continue by using a numerical solver. Below is the explanation of the approach used.

If $(\varphi, v)(s; \varphi_0, v_0)$ solves (3.14) then

$$(a, b)(s; \varphi_0, v_0) = \left(\dot{\varphi}, \frac{\varphi}{s} \right)(s; \varphi_0, v_0)$$

solves (3.4) and satisfies

$$\lim_{s \rightarrow 0_+} sb(s) = \varphi_0, \quad \lim_{s \rightarrow 0_+} (ab^{d-1})(s) = v_0.$$

Moreover, in view of (H3),

$$c_0 := \lim_{s \rightarrow 0_+} \frac{d}{ds} \left(\dot{\varphi} \left(\frac{\varphi}{s} \right)^{d-1} \right) = \lim_{s \rightarrow 0_+} \dot{v}(s) = \frac{(d-1)\gamma_0}{\varphi_0 h''(v_0)}, \quad \gamma_0 = \begin{cases} \gamma, & d \geq 3, \\ \gamma - g'(0), & d = 2. \end{cases}$$

Thus

$$\frac{d}{ds}(\varphi^d(s)) = d[\dot{\varphi}\varphi^{d-1}](s) = d(v_0s^{d-1} + c_0s^d + o(s^d)) \quad \text{for } s \ll 1 \tag{7.16}$$

and hence

$$\varphi(s) = \sqrt[d]{\varphi_0^d + v_0s^d + \left(\frac{d}{d+1} \right) c_0s^{d+1} + o(s^d)} \quad \text{for } s \ll 1. \tag{7.17}$$

Given $\varphi_0, v_0 > 0$, we construct the solution $(\varphi, v)(s; \varphi_0, v_0)$ as follows: We pick a sufficiently small $s_0 > 0$ and select the approximate values at $s = s_0 > 0$ (following (7.16), (7.17)) by

$$\widehat{b}(s_0) = \frac{1}{s_0} \sqrt[d]{\varphi_0^d + v_0 s_0^d + \left(\frac{d}{d+1}\right) c_0 s_0^{d+1}},$$

$$\widehat{a}(s_0) = (v_0 s_0^{d-1} + c_0 s_0^d) (s_0 \widehat{b}(s_0))^{1-d}.$$

We use these as initial data at s_0 and then solve numerically (3.4) (using the standard MATLAB solver `ode15`) on the interval $[s_0, \widehat{T}]$, where \widehat{T} is the maximal interval of existence of the approximate solution $(\widehat{a}, \widehat{b})$. At $s = \widehat{T}$ computations break down due to the singularity $Q = 0$.

We construct the dynamic bifurcation curve in Fig. 2 as follows. For a stored energy Φ with $g(x) = \frac{1}{2}x^2, h(x) = (x - 1) \ln(x)$, we fix $v_0 = H$, where $H > 0$ is the unique number that satisfies $h'(H) = 0$ corresponding to a stress free cavity. Then, we pick $\varphi_0 \in [0.05, 2.7]$ and $s_0 > 0$ and compute the numerical solution

$$\{(\widehat{a}_n, \widehat{b}_n)(s_0, \varphi_0, H)\}_{n=0}^N, \quad \text{on the mesh } s_0 < s_1 < \dots < s_N = \widehat{T},$$

that approximates $(\widehat{a}, \widehat{b})(s; s_0, \varphi_0, H)$ as described in the previous paragraph. Finally, we determine the point $s_{n_*} \in (s_0, s_N]$ that best fits the condition

$$s_{n_*} \approx \sqrt{\frac{\Phi_1(\widehat{a}_{n_*}, \widehat{b}_{n_*}) - \Phi_1(\widehat{b}_{n_*}, \widehat{b}_{n_*})}{\widehat{a}_{n_*} - \widehat{b}_{n_*}}} \approx \sigma(\varphi_0, V(\varphi_0)),$$

which corresponds to the Rankine-Hugoniot condition. This in turn provides the approximate value of the stretching

$$\Lambda(\varphi_0, H) \approx \widehat{b}_{n_*}(s_0, \varphi_0, H).$$

This procedure is repeated for a sequence of values φ_0 values in the interval $[0.05, 2.7]$ and gives the dynamic bifurcation curve in Fig. 2.

The bifurcation curve for elastostatics (corresponding to the boundary value problem (5.44) in Sect. 5.3.2) is constructed in an analogous fashion. The only difference is that solutions are now computed for the modified system (3.4) (obtained by replacing the term $s^2 - \Phi_{11}$ in (3.4) by the term $-\Phi_{11}$) on the interval $(s_0, 1]$. The value $\widehat{b}(1; s_0, \varphi_0, H)$ then gives the stretching $\lambda(\varphi_0, H)$ at the boundary of the unit ball. The curve in Fig. 2 is the graph of $\lambda(\varphi_0, H)$ with $\varphi_0 \in [0.05, 2.7]$.

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