AN APPROXIMATE TECHNIQUE FOR ESTIMATING EIGENVALUES AND EIGENFUNCTIONS OF THE ACOUSTIC FIELD IN SHALLOW WATER OVER AN ELASTIC SEA-BED

M.I. Taroudakis University of Crete and FORTH/IACM, 700 13 Heraklion, Greece
J.K. Mastrokalos University of Crete and FORTH/IACM, 700 13 Heraklion, Greece

1 INTRODUCTION

An alternative method for the calculation of the eigenvalues and associated eigenfunctions of the "depth problem" associated with the normal-mode solution of the acoustic propagation in shallow water over a homogeneous elastic sea-bed is presented. The work can be viewed as an amendment to previous works related to the same subject under the additional condition that the sound speed profile in the water column varies with depth. In the present work, a finite-difference scheme is used for the numerical treatment of the eigenvalue/eigenfunction problem under an iterative procedure based on the Inverse Power Iteration method, with starting eigenvalues those of a constant sound speed profile. The Effective Depth method is used for the estimation of the complex eigenvalues for the environment with constant sound speed profile. The details of the method are presented in section 2.2. Although we started the study under the objective of obtaining approximate eigenvalues for the problem of varying with depth sound speed profile, the iterative technique came out to provide eigenvalues and eigenfunctions and eventually the pressure field with accuracy comparable to that of the KRAKEN-C program as illustrated by the test cases studied.

2 ESTIMATION OF EIGENVALUES-EIGENFUNCTIONS

2.1 The Depth Problem

The equation governing the underwater sound propagation problem due to a monochromatic point source located at $(r,z)=(0,z_0)$ of unit strength in range-independent environment is the Helmholtz equation, which in cylindrical coordinates with axial symmetry is written as:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right)+\rho(z)\frac{\partial}{\partial z}\left(\frac{1}{\rho(z)}\frac{\partial p}{\partial z}\right)+k^2(z)\,p=-\frac{\delta(z-z_0)\delta(r)}{2\pi r} \tag{1}$$

where $p(r,z)$ is the acoustic pressure field, $k(z)=\omega/c(z)$ is the wave number, $\omega=2\pi f$ is the angular frequecy, $c(z)$ is the velocity of sound propagation and $\rho(z)$ is the water density. This equation associated with the conditions of a pressure release water-air interface, an appropriate boundary condition at the interface between water and seabed and a Sommerfeld radiation condition for the behavior of the pressure field at infinity defines the well posed problem of acoustic wave propagation in a waveguide.

Expressing the solution of the problem (the Green's function) in series expansion we get

$$p(r,z)=\sum A_n(r)\,u_n(z) \tag{2}$$

where $A_n(r)$ is the range dependence of the Green's function, and $u_n(z)$ is the eigenfunction defined by the "Depth Problem"

$$\frac{d^2 u_n}{dz^2}(z)+\left(k^2(z)-\lambda_n^2\right)u_n(z)=0\ ,\ 0\leq z\leq D \tag{3.a}$$

$$u_n(0)=0 \tag{3.b}$$

$$\frac{u_n(D)}{u_n'(D)}=I(\lambda_n) \tag{3.c}$$

where D is the water depth and $I(\lambda_n)$ is an appropriate impedance condition which associates the eigenfunction to its derivative at depth $z=D$. We will discuss the form of the impedance condition in section 2.2.

This is a well known problem in underwater acoustic propagation modelling and several approaches have been proposed for its treatment[1-3]. For a sound speed profile which does not change with depth, an analytic treatment of the characteristic equation leading to the calculation of the eigenvalues has been presented in[4] while the full solution of the eigenvalue-eigenfunction problem using the method of the effective depth[5] has been used by the authors in the development of their propagation models treating the problem of acoustic field prediction over an elastic halfspace.

For sound speed profile varying with depth we studied an iterative technique based on initial assumptions of the complex eigenvalue vector and aiming at a good approximation of the true eigenvalue-eigenfunction system. It came out that this scheme converges to the true eigenvalues of the "depth" problem thus consisting an alternative way for treating the problem. This iterative scheme is presented in the next section.

2.2 Calculation of the eigenvalues by means of an iterative scheme

Problem (3) is solved by means of a finite-difference scheme. We discretize the interval $[0,D]$ uniformly into N intervals of length h so that

$$z_j=jh\ ,\ j=0,1,2,\ldots,N \tag{4.a}$$

$$u_n^j=u_n(z_j)\ ,\ u_n^0=0 \tag{4.b}$$

$$k_j^2=k^2(z_j) \tag{4.c}$$

For the corresponding discrete problem we approximate the second derivative of $u_n(z)$ by means of a second order finite difference :

$$\frac{d^2 u_n(z_j)}{dz^2}\approx\frac{u_n^{j-1}-2\mathrm{u}_n^j+u_n^{j+1}}{h^2}\ ,\ j=1,\ldots,N-1 \tag{5}$$

Thus, equation (3.a) becomes

$$\frac{u_n^{j-1}-2\mathrm{u}_n^j+u_n^{j+1}}{h^2}+\left(k_j^2-\lambda_n^2\right)u_n^j=0\ ,\ j=1,\ldots,N-1 \tag{6}$$

In order to define a well-posed linear system of equations we need an additional independent equation which is derived using (3.a), (3.c) and Taylor expansion of u_n at $z-h$. The N^{th} equation is

$$2\mathrm{u}_n^{N-1}+\left[h^2 k_N^2-2+\frac{2\mathrm{h}}{I(\lambda_n)}\right]u_n^N=h^2\lambda_n^2 u_n^N \tag{7}$$

Thus the linear system is written, in matrix form

$$\frac{1}{h^2}\begin{bmatrix} h^2k_1^2-2 & 1 & 0 & \cdots & 0 \\ 1 & h^2k_2^2-2 & 1 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & \cdots & 1 & h^2k_{N-1}^2-2 & 1 \\ 0 & 0 & \cdots & 2 & h^2k_N^2-2+\frac{2h}{I(\lambda_n)} \end{bmatrix}\begin{bmatrix} u_n^1 \\ u_n^2 \\ \vdots \\ u_n^{N-1} \\ u_n^N \end{bmatrix} = \lambda_n^2 \begin{bmatrix} u_n^1 \\ u_n^2 \\ \vdots \\ u_n^{N-1} \\ u_n^N \end{bmatrix} \tag{8}$$

where the $N\times N$ matrix is denoted by $A(\lambda_n)$.

Given the eigenvalues λ_n , the system is solved for the unknown values of u_n^j by setting $u_n^N=1$, using backward computation for u_n^j , $j=N-1,\ldots,1$ and then applying the appropriate normalization condition[1].

In order to obtain the eigenvalues we need additional consideration for the impedance function $I(\lambda_n)$. Considering that in an infinitesimal interval, $(D-\varepsilon, D)$, $c(z)$ is constant, $u_n(z)$ has the form of an exponential function and the impedance function can be formulated[1] as

$$I(\lambda_n)=\frac{(1+R(\lambda_n))}{i\gamma_{1n}(1-R(\lambda_n))} \tag{9}$$

where $\gamma_{1n}^2(z)=k^2(z)-\lambda_n^2$ and $R(\lambda_n)$ is the plane wave reflection coefficient between water and the seabed.

We propose to determine the eigenvalues using an iterative scheme which is based on the Inverse Power Iteration method[6].

The eigenvalues are those λ_n for which system (8) is singular, thus they must satisfy

$$det\left(A(\lambda_n)-\lambda_n^2 I\right)=0 \tag{10}$$

Given an initial approximation μ of an eigenvalue λ for a constant matrix A , Inverse Power method iterates[5] $\boldsymbol{v}^{\boldsymbol{k+1}}=(A-\mu I)^{-1}\boldsymbol{v}^{\boldsymbol{k}}$ until $\boldsymbol{v}^{\boldsymbol{k}}\rightarrow\boldsymbol{v}$, with $\|\boldsymbol{v}^{\boldsymbol{0}}\|=1$ an initial vector and $\boldsymbol{v}$ the eigenvector of A which is associated to the closest eigenvalue to μ and is calculated by formula

$$\lambda=\mu+\frac{\boldsymbol{v}^{\boldsymbol{H}}A\,\boldsymbol{v}}{\boldsymbol{v}^{\boldsymbol{H}}\boldsymbol{v}} \tag{11}$$

where the superscript H indicates the conjugate transpose.

Since the finite difference problem matrix depends on λ_n , $A=A(\lambda_n)$, we suggest after having obtained an approximation λ_n^1 by inverse power iteration to compute a new finite difference matrix $A(\lambda_n^1)$ on which we apply Inverse Iteration again and then continue consecutively until the sequence $\{\lambda_n^i\}_i$ converges.

In our scheme, as initial approximations of the eigenvalues, we have chosen to use the eigenvalues of a constant sound speed profile close to the actual one, using the effective-depth method.

In brief, the iterative scheme suggested in this work is as follows :

1. $\mu_n = effectiveDepth(c_0, [ENV])$, $c_0 = const.$
2. $\lambda_n^0 \leftarrow \mu_n$
3. until $\{\lambda_n^i\}_i$ $converges$
4. $\lambda_n^{i+1} = InvPowIt(A(\lambda_n^i))$, $i = 0,1,2,\ldots$

where $[ENV]$ indicates the environmental parameters.

Having studied the the application of the above scheme in waveguides with different sound speed profiles and environmental parameters, several issues have to be stressed out :

- We apply the Effective Depth scheme for the initial assumption of the eigenvalues with $c_0 = min\{c(z)\}$. This choice guarantees computation of low order modes which otherwise could be omitted.
- The suggested scheme may generate duplicates due to the fact that some of the initial assumptions λ_n^0 , $n = 1,2,3,\ldots$ may be too close to each other and the iteration leads to the same limit. It is obvious that these duplicates should be excluded from the computation of the pressure field
- It is still possible that the iterative scheme omits some eigenvalues. We overcome this by applying a high to low reordering of eigenvalues based on their real part and taking the mean value between two adjacent eigenvalues. This value is then used as an initial approximation to the suggested iterative scheme.

3 Applications

As an illustration of the effectiveness of the technique described above we approximate the eigenvalues and discrete values of the associated eigenfunctions of the depth problem for a waveguide of depth $D = 800\,m$ and for two different sound speed profiles in water ($c_1(z)$). The characteristic parameters of the underwater environment are $\rho_1 = 1000\,kg/m^3$, $\rho_2 = 1300\,kg/m^3$ the density in water and in bottom respectively, $c_2 = 1700\,m/sec$ the speed of compressional waves in bottom, $c_s = 500\,m/sec$ the shear wave speed in bottom and the source frequency f is $50\,Hz$. The profiles considered are presented in table (1):

	z (m)	*$c_1(z)$ (m/sec)*
(i)	*0* *200* *800*	*1500* *1490* *1550*
(ii)	*0* *800*	*1490* *1550*

Table 1: Sound speed profiles (i) and (ii) of our test case

In tables (2) and (3) we present the first ten complex eigenvalues computed by the suggested iterative scheme and compare them to those computed by KRAKEN-C while the first five associated eigenfunctions are shown in figures (1) and (2), for sound speed profiles (i) and (ii) respectively. As an additional test we also compare the transmission loss typically defined as

$$TL(r,z) = -20\log_{10}\left|\frac{p(r,z)}{1/4\pi}\right| \qquad (12)$$

using our code in comparison with KRAKEN-C code. Calculation of the pressure field in our code is based on formula (2) where the range dependence is given by $A_n(r)=\frac{i}{4}u_n(z_0)H_0^{(1)}(\lambda_n r)$ and z_0 denotes the source depth. Substituting in (2) the pressure field is computed [1,3]

$$p(r,z)=\frac{i}{4}\sum u_n(z_0)u_n(z)H_0^{(1)}(\lambda_n r) \quad (13)$$

Transmission loss was calculated along a $2\,km$ range in the horizontal direction at the source depth which was set at $z_0=560\,m$. Figures (3) and (4) correspond to sound speed profiles (i) and (ii) respectively. The blue line represents transmission loss calculated by KRAKENC and red dots represent results by the technique described in this work.

Mode	Eigenvalue	Eigenvalue (KRAKENC)
1	2.1022314428e-01 + i6.2174931868e-24	2.1022314430e-01 + i9.0422924060e-19
2	2.0933811819e-01 + i4.4904172867e-21	2.0933811820e-01 + i9.6491314070e-19
3	2.0852967384e-01 + i2.0872898436e-18	2.0852967380e-01 + i1.4812774060e-18
4	2.0770369973e-01 + i6.2295342709e-16	2.0770369970e-01 + i6.2275176490e-16
5	2.0692070468e-01 + i9.2454406721e-14	2.0692070470e-01 + i9.2454631190e-14
6	2.0615986383e-01 + i7.8046788180e-12	2.0615986380e-01 + i7.8046788790e-12
7	2.0542436009e-01 + i3.5585194224e-10	2.0542436010e-01 + i3.5585194410e-10
8	2.0471831547e-01 + i8.4815702598e-09	2.0471831550e-01 + i8.4815702610e-09
9	2.0403682959e-01 + i1.0319159603e-07	2.0403682960e-01 + i1.0319159600e-07
10	2.0336209721e-01 + i6.1701795904e-07	2.0336209720e-01 + i6.1701786760e-07

Table 2: The first ten eigenvalues for sound speed profile (i) computed with the suggested technique and KRAKENC.

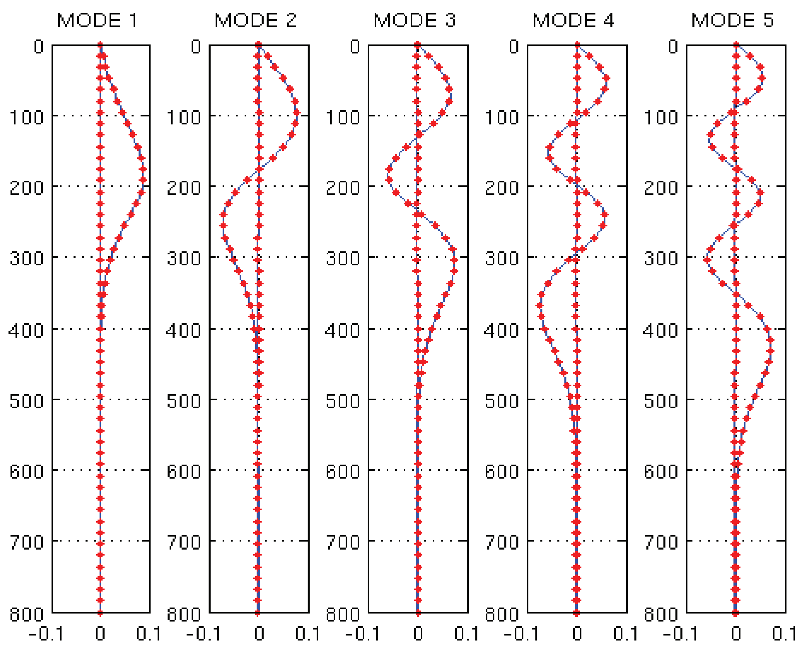

Figure 1: The first five eigenfunctions for sound speed profile (i). KRAKENC (blue line), suggested scheme (red dots)

Mode	Eigenvalue	Eigenvalue (KRAKENC)
1	2.093426382e-01 + i7.252119530e-22	2.093426382e-01 + i-1.746304611e-19
2	2.082224358e-01 + i3.254951542e-21	2.082224358e-01 + i-2.727871980e-19
3	2.073080401e-01 + i1.054164040e-17	2.073080401e-01 + i1.073436848e-17
4	2.065019332e-01 + i7.190781281e-15	2.065019332e-01 + i7.190817661e-15
5	2.057667208e-01 + i1.557030720e-12	2.057667208e-01 + i1.557027657e-12
6	2.050830694e-01 + i1.322326178e-10	2.050830694e-01 + i1.322326161e-10
7	2.044392579e-01 + i4.847987550e-09	2.044392579e-01 + i4.847987552e-09
8	2.038258563e-01 + i7.778418974e-08	2.038258563e-01 + i7.778418997e-08
9	2.032239773e-01 + i5.332898405e-07	2.032239773e-01 + i5.332897781e-07
10	2.025905240e-01 + i1.715131411e-06	2.025905240e-01 + i1.715131576e-06

Table 3: The first ten eigenvalues for sound speed profile (ii) computed with the suggested technique and KRAKENC.

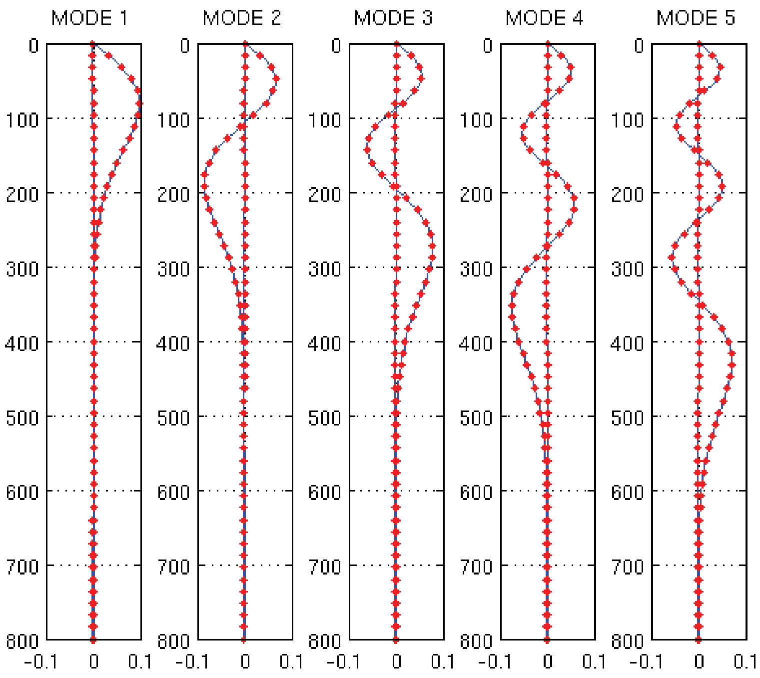

Figure 2: The first five eigenfunctions for sound speed profile (ii). KRAKENC (blue line), suggested scheme (red dots)

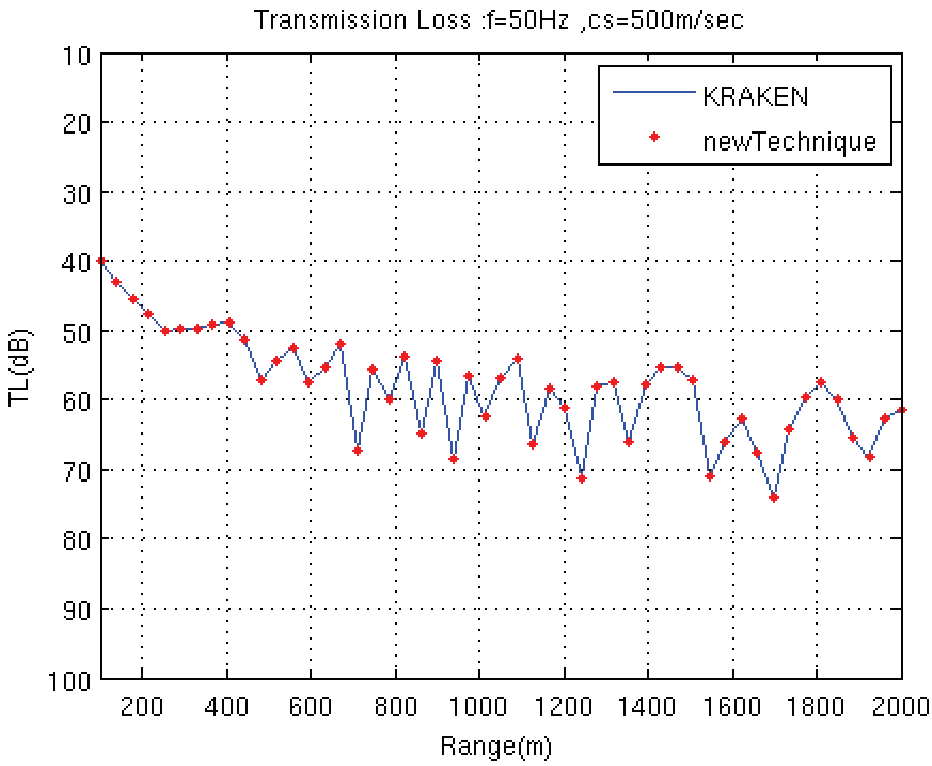

Figure 3: Comparison of transmission loss for sound speed profile (i)

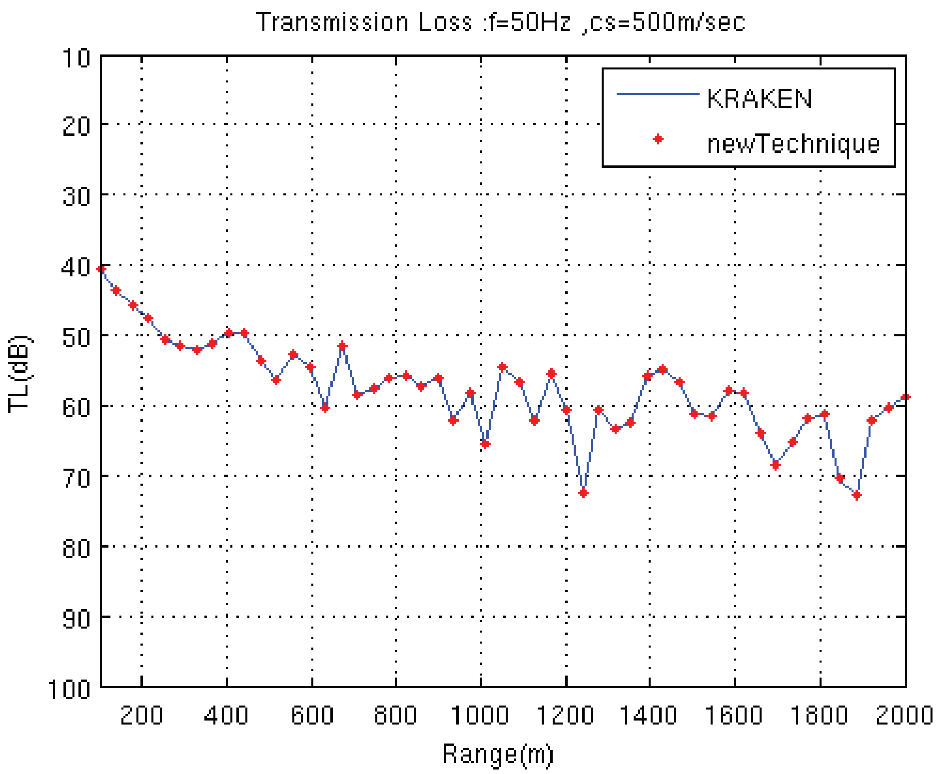

Figure 4: Comparison of transmission loss for sound speed profile (ii)

It is obvious that both the eigenvalue-eigenfunction and the pressure field calculations give excellent correspondence between KRAKEN-C and our scheme. Convergence of our scheme in both cases is ensured after less than ten iterations.

4 Conclusions

An iterative scheme for the calculation of the eigenvalues and the associated eigenfunctions of the “depth problem” of acoustic propagation in a shallow water range-independent waveguide with sound speed in water varying with depth, over a homogeneous elastic half-space has been presented. A finite-difference scheme has been applied in connection with the appropriate impedance function in the water-seabed interface. The eigenvalue matrix problem has been formulated in terms of an iterative scheme. The initial estimations of the eigenvalues are taken using an effective-depth approach for a constant sound speed profile corresponding to the minimum value of the actual profile. Several considerations have been taken into account to ensure that the scheme converges to the whole set of propagating modes. By means of test cases it has been shown that although the approach has been studied on the purpose of providing approximate eigenvalues for the calculation of the acoustic field for practical applications, the iterative scheme suggested here gives excellent results at least for the cases studied in comparison with the output of the KRAKEN-C program. Thus it can be viewed as an alternative way to provide eigenvalues and eigenfunctions for the series expansion of the acoustic field in a shallow water waveguide over an elastic homogeneous half-space.

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